CHAPTER

R Elementary Algebra Review

The image to the left of a sun carrier shell fossil (*Stellaria solaris*) demonstrates how a specific sequence of numbers, called the Fibonacci sequence, occurs frequently in nature. Problem 92 in Section R.2 explores the Fibonacci sequence further.

The Big Picture: Putting It Together

As the "R" in the title implies, this chapter is a review of Elementary Algebra. The purpose of this chapter is to help you recall mathematical concepts that you learned in earlier courses. The topics in this chapter are important building blocks that will help you succeed in this course.

Outline

R.1 Sets and Classification of Numbers

Objectives

1. Use Set Notation
2. Classify Numbers
3. Approximate Decimals by Rounding or Truncating
4. Plot Points on the Real Number Line
5. Use Inequalities to Order Real Numbers

1 Use Set Notation

A **set** is a well-defined collection of objects. "Well-defined" means that there is a rule for determining whether a given object is in the set. For example, the students enrolled in Intermediate Algebra at your college is a set. The collection of numbers 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9 may also be identified as a set. If D represents this set of numbers, then

$$D = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

This notation uses braces $\{ \quad \}$ to enclose the objects, or **elements,** in the set. This method of representing a set is called the **roster method.**

EXAMPLE 1 Using the Roster Method

Write the set that represents the vowels.

Classroom Example 1
Write the set that represents the last three letters of the alphabet.
Answer: {x, y, z}

Solution

The vowels are a, e, i, o, and u, so write

$$V = \{a, e, i, o, u\}$$

In Other Words
The symbol "|" means "such that", so read the set to the right as "D is the set of all x such that x is a digit."

Another way to denote a set is to use **set-builder notation.** The numbers in the set $D = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ are called digits. Using set-builder notation, the set D of digits can be written as

$$D = \{x \mid x \text{ is a digit}\}$$

In algebra, letters such as x, y, a, b, and c are used to represent numbers. When the letter can be any number in a set of numbers, it is called a **variable.** In the set D, the letter x can represent any digit, so x is a variable that can take on the value 0, 1, 2, 3, 4, 5, 6, 7, 8, or 9.

EXAMPLE 2 Using Set-Builder Notation

Use set-builder notation to represent the following sets.

(a) The set of all even digits
(b) The set of all odd digits

Classroom Example 2
Use set-builder notation to represent the following sets.
(a) The set of all digits greater than 2
(b) The set of all digits less than 8
Answer:
(a) $A = \{x \mid x \text{ is a digit greater than } 2\}$
(b) $B = \{x \mid x \text{ is a digit less than } 8\}$

Solution

(a) Let E represent the set of all even digits, so that

$$E = \{x \mid x \text{ is an even digit}\}$$

(b) Let O represent the set of all odd digits, so that

$$O = \{x \mid x \text{ is an odd digit}\}$$

Quick ✓

1. A set is a well-defined collection of objects.
2. The objects in a set are called elements.

In Problems 3 and 4, use set-builder notation and the roster method to represent each set.

3. The set of all digits less than 5 $\{x|x \text{ is a digit less than } 5\}; \{0, 1, 2, 3, 4\}$
4. The set of all digits greater than or equal to 6
   $\{x|x \text{ is a digit greater than or equal to } 6\}; \{6, 7, 8, 9\}$

Elements in sets are never listed more than once. For example, do not write $\{1, 2, 3, 2\}$; write $\{1, 2, 3\}$. Also, the order in which the elements are listed does not matter. For example, $\{2, 3\}$ and $\{3, 2\}$ represent the same set.

More notation for describing sets will now be introduced.

Teaching Tip
Emphasize the difference between $\subset$ and $\subseteq$. You can relate this notation to inequality notation.

Work Smart
The empty set is represented as either $\varnothing$ or $\{\ \}$. Never write $\{\varnothing\}$ to represent the empty set because this notation means a set that contains the empty set.

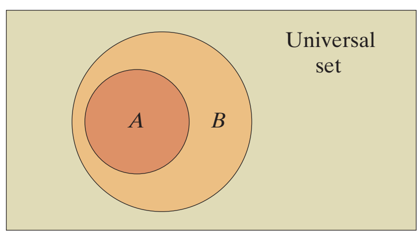
Set Notation

- If two sets A and B have the same elements, then A **equals** B, written $A = B$.
- If every element of a set A is also an element of a set B, then A is a **subset** of B, written $A \subseteq B$.
- If $A \subseteq B$ and $A \neq B$, then A is a **proper subset** of B, written $A \subset B$. Put another way, A is a proper subset of B if all elements in A are also in B and there are elements in B that are not in A.
- If a set A has no elements, it is called the **empty set,** or **null set,** and is denoted by the symbol $\varnothing$ or $\{\ \}$. The empty set is a subset of every set; that is $\varnothing \subseteq A$ for any set A.

When working with sets, we usually designate a **universal set,** which is the set of all elements of interest to us. For instance, in Example 2, we were interested in the set of all digits, so the universal set is the set of all digits.

It is often helpful to draw pictures of sets because the pictures help us visualize relations among sets. Pictures of sets are called **Venn diagrams,** in honor of John Venn (1834 – 1923). In Venn diagrams, sets are represented as circles enclosed in a rectangle. The rectangle represents the universal set. For example, if $A = \{1, 2\}$, $B = \{1, 2, 3, 4, 5\}$, and the universal set is $U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, then $A \subset B$. Figure 1 illustrates the relation between sets A and B in a Venn diagram.

Figure 1
Venn diagram with $A \subset B$.



EXAMPLE 3

Using Set Notation

Classroom Example 3
Let $A = \{2, 4, 6, 8\}$, $B = \{1, 2, 3, 4, 5\}$, $C = \{2, 3, 4\}$, and $D = \{4, 6\}$.
Write True or False for each statement.
(a) $D \subseteq A$ (b) $D \subseteq B$
(c) $C \subset B$ (d) $\emptyset \subseteq C$
(e) $B = C$
Answer: (a) True (b) False (c) True (d) True (e) False

Let $A = \{0, 1, 2, 3, 4, 5\}$, $B = \{3, 4, 5\}$, $C = \{5, 4, 3\}$, and $D = \{3, 4, 5, 6\}$. Write True or False for each statement.

(a) $B \subseteq A$ **(b)** $D \subseteq A$ **(c)** $B = C$
(d) $C = D$ **(e)** $B \subset C$ **(f)** $\emptyset \subseteq C$

Solution

(a) The statement $B \subseteq A$ is True because all the elements in B are also elements in A.

(b) The statement $D \subseteq A$ is False because there is an element in D, 6, that is not in A.

(c) The statement $B = C$ is True because sets B and C have the same elements.

(d) The statement $C = D$ is False because there is an element in D, 6, that is not in C.

(e) In order for B to be a proper subset of C, it must be the case that all elements in B are also elements in C. In addition, there must be at least one element in C that is not in B. Because $B = C$, the statement $B \subset C$ is False. Note, however, that $B \subseteq C$ is True and $B \subset D$ is also True.

(f) The statement $\emptyset \subseteq C$ is True because the empty set is a subset of every set. ●

Quick ✓

5. *True or False* The order in which elements are listed in a set does not matter. True

6. If every element of a set A is also an element of a set B, then A is a subset of B, written $A \subseteq B$.

7. *True or False* If a set has no elements, it is called the empty set and is denoted $\{\emptyset\}$. False

In Problems 8–11, let $A = \{a, b, c, d, e, f, g\}$, $B = \{a, b, c\}$, $C = \{c, d\}$, and $D = \{c, b, a\}$. Write True or False for each statement. Be sure to justify your answer.

8. $B \subseteq A$ True **9.** $B = C$ False **10.** $B \subset D$ False **11.** $\emptyset \subseteq A$ True

The symbol $\in$ (which is read "is an element of") is used to indicate that a particular element is in a set. For example, $7 \in \{1, 3, 5, 7, 9\}$ means "7 is an element of the set $\{1, 3, 5, 7, 9\}$." If an element is not in a set, use the symbol $\notin$ (which is read "is not an element of"). For example, write "b is not a vowel" as $b \notin \{a, e, i, o, u\}$.

EXAMPLE 4

Using Set Notation

Classroom Example 4
Write True or False for each statement.
(a) $5 \in \{1, 2, 3, 4, 5\}$
(b) $-2 \notin \{2, 4, 6\}$
(c) Ohio $\in \{x \mid x$ is a U.S. state that borders the Pacific Ocean$\}$
Answer: (a) True (b) True (c) False

Write True or False for each statement.

(a) $3 \in \{x \mid x \text{ is a digit}\}$

(b) $h \notin \{a, e, i, o, u\}$

(c) $\dfrac{1}{2} \in \left\{x \,\middle|\, x = \dfrac{p}{q}, \text{ where } p \text{ and } q \text{ are digits, } q \neq 0\right\}$

Solution

(a) The statement $3 \in \{x \mid x \text{ is a digit}\}$ is True because 3 is a digit.

(b) The statement $h \notin \{a, e, i, o, u\}$ is True because h is not an element of the set $\{a, e, i, o, u\}$.

Work Smart

Using correct notation is important:
$3 \subset \{1, 2, 3\}$ is incorrect.
$3 \in \{1, 2, 3\}$ is correct.

(c) The statement $\frac{1}{2} \in \left\{x \middle| x = \frac{p}{q}, \text{ where } p \text{ and } q \text{ are digits, } q \neq 0\right\}$ is True because $\frac{1}{2}$ is of the form $\frac{p}{q}$, where $p = 1$ and $q = 2$.

Quick ✓

In Problems 12–14, answer True or False for each statement.

12. $5 \in \{0, 1, 2, 3, 4, 5\}$ True

13. Michigan $\notin$ {Illinois, Indiana, Michigan, Wisconsin} False

14. $\frac{8}{3} \in \left\{x \middle| x = \frac{p}{q}, \text{ where } p \text{ and } q \text{ are digits, } q \neq 0\right\}$ True

Summary Set Notation

$A = B$	means	set A and set B have the same elements.
$A \subseteq B$	means	all the elements in set A are also elements in set B.
$A \subset B$	means	all the elements in set A are also elements in set B, but there is at least one element in set B that is not in set A.
$\varnothing$ or $\{\ \}$	means	the empty set. The empty set has no elements.
$5 \in A$	means	5 is an element in set A.
$5 \notin A$	means	5 is not an element in set A.

2 Classify Numbers

Sets are discussed because it is helpful to classify the various kinds of numbers that we deal with as sets.

Definition

The **natural numbers,** or **counting numbers,** are the numbers in the set $\mathbb{N} = \{1, 2, 3, \ldots\}$.

The three dots in the definition above, called an **ellipsis,** indicate that the pattern continues indefinitely. The counting numbers are often used to count things. For example, we can count the cars that arrive at a McDonald's drive-through, or we can count the letters in the alphabet.

Suppose you had \$7 and purchased a drink and a hot dog at a baseball game for \$7. Can the counting numbers be used to describe the amount of money you have left? No, because zero is not part of the natural numbers. A new number system is needed to describe the remaining amount. We need the *whole number system*.

Definition

The **whole numbers** are the numbers in the set $W = \{0, 1, 2, 3, \ldots\}$.

The whole numbers consist of the set of counting numbers together with the number 0, so $\mathbb{N} \subset W$.

Now suppose you have a balance of \$100 in your checking account and you write a check for \$120. Can the whole numbers be used to describe your new balance? No, because the set of whole numbers does not contain negative numbers. You need the *integers*.

Definition

The **integers** are the numbers in the set $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.

In Other Words

The set of natural numbers is a subset of the set of whole numbers. The set of whole numbers is a subset of the set of integers.

Notice that the set of natural numbers is a proper subset of the set of whole numbers, and the set of whole numbers is a proper subset of the set of integers. That is, $\mathbb{N} \subset W$ and $W \subset \mathbb{Z}$. As the number system is expanded, new and usually more complicated problems can be discussed. For example, the whole numbers allow the discussion of the absence of something because they include zero, but the counting numbers do not. The integers allow us to deal with problems involving both negative and positive quantities, such as profit (positive counting numbers) and loss (negative counting numbers).

To represent a portion of a dollar or a portion of a whole pie, our number system is enlarged to include *rational numbers*.

In Other Words

A rational number is a number that can be expressed as a fraction, where the numerator is any integer and the denominator is a nonzero integer.

Definition

A **rational number** is a number that can be expressed as a quotient, $\frac{p}{q}$, of two integers. The integer p is the **numerator,** and the integer q, which cannot be 0, is the **denominator.** The set of rational numbers are the numbers

$$\mathbb{Q} = \left\{x \,\middle|\, x = \frac{p}{q}, \text{ where } p \text{ and } q \text{ are integers, } q \neq 0\right\}.$$

Examples of rational numbers are $\frac{3}{4}, \frac{4}{3}, \frac{0}{6}, -\frac{4}{5}$, and $\frac{33}{5}$. Because $\frac{p}{1} = p$ for any integer p, it follows that the set of integers is a proper subset of the set of rational numbers $(\mathbb{Z} \subset \mathbb{Q})$. For example, 5 is a rational number because it can be written as $\frac{5}{1}$, but more specifically, it is an integer. More specifically than that, it is a counting number.

Rational numbers can also be represented as *decimals*. The **decimal** representation of a rational number is found by carrying out the division indicated. For example,

$$\frac{4}{5} = 0.8 \qquad \frac{7}{2} = 3.5 \qquad -\frac{2}{3} = -0.6666\ldots = -0.\overline{6} \qquad \frac{2}{11} = 0.181818\ldots = 0.\overline{18}$$

Work Smart

To write $\frac{2}{11}$ as a decimal, write $11\overline{)2}$ and carry out the division as follows:

$$\begin{array}{r} 0.181 \\ 11\overline{)2.000} \\ \underline{11} \\ 90 \\ \underline{88} \\ 20 \\ \underline{11} \\ 9 \end{array}$$

and so on to get $0.1818\ldots = 0.\overline{18}$.

Notice the line above the 6 in $-0.\overline{6}$. This **repeat bar** represents the fact that the pattern continues. Similarly, $0.\overline{18}$ means the block of numbers 18 will continue indefinitely to the right of the decimal point.

Every rational number may be represented by a decimal that either **terminates** $\left(\text{as in the case of } \frac{4}{5} = 0.8 \text{ and } \frac{7}{2} = 3.5\right)$ or is **nonterminating** with a repeating block of decimals $\left(\text{as in the case of } -\frac{2}{3} = -0.\overline{6} \text{ and } \frac{2}{11} = 0.\overline{18}\right)$.

What if a decimal neither terminates nor has a block of digits that repeat? These decimals represent a set of numbers called *irrational numbers.* Every **irrational number** may be represented by a decimal that neither terminates nor repeats. This means that irrational numbers cannot be written in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$. An example of an irrational number is π, whose value is approximately 3.14159265359. Another example of an irrational number is $\sqrt{2}$. This is the number whose square is 2. Its value is approximately 1.414213562.

Work Smart

Although it is true that $\sqrt{2}$ is an irrational number, not all numbers involving the $\sqrt{}$ symbol are irrational. For example, $\sqrt{4} = 2$, is a rational number.

Definition

Together, the set of rational numbers and the set of irrational numbers form the set of **real numbers.** The set of real numbers is denoted using the symbol $\mathbb{R}$.

Teaching Tip

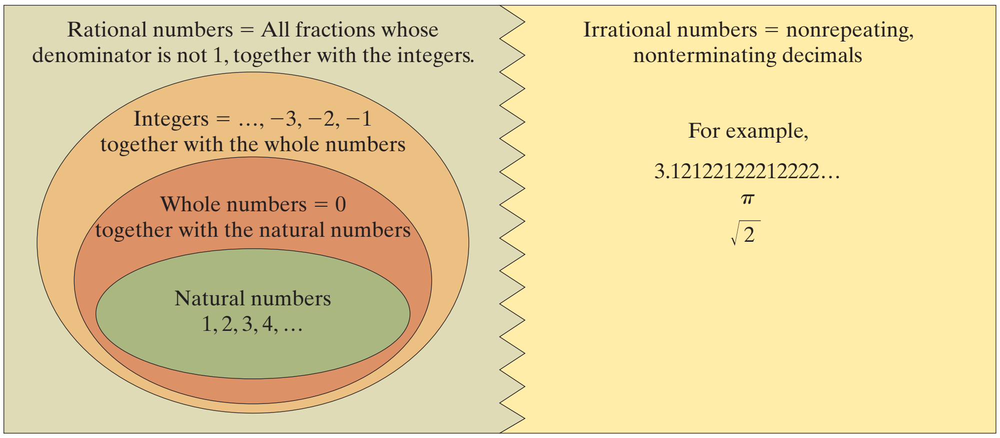
Learning the relationship among the sets that make up the real number system is difficult for students. Use Figure 2 to show the structure. Use a table and have students check each category to which a number belongs.

	-2
Natural	
Whole	
Integer	✓
Rational	✓
Irrational	
Real	✓

Figure 2 shows the relationships among the various types of numbers. The universal set in Figure 2 is the set of all real numbers. It is extremely important to be able to distinguish among the various number systems—in particular, it is important to know the difference between a rational number and an irrational number. Notice that if a number is rational, it cannot be irrational, and vice versa.

Figure 2

The Set of Real Numbers



EXAMPLE 5 Classifying Numbers in a Set

List the numbers in the set

$$\left\{3, -\frac{3}{5}, -1, 0, \sqrt{2}, 5.\overline{94}, 4.2122122212222\ldots, -\frac{8}{2}, \frac{\pi}{2}\right\}$$

that are

(a) Natural numbers **(b)** Whole numbers **(c)** Integers
(d) Rational numbers **(e)** Irrational numbers **(f)** Real numbers

Solution

(a) 3 is the only natural number.

(b) 0 and 3 are the whole numbers.

(c) $3, -1, 0$, and $-\frac{8}{2}$ are the integers $\left(-\frac{8}{2}\text{ is an integer because it simplifies to }-4\right)$.

(d) $3, -\frac{3}{5}, -1, 0, 5.\overline{94}$, and $-\frac{8}{2}$ are the rational numbers.

(e) $\sqrt{2}$, $4.2122122212222\ldots$, and $\frac{\pi}{2}$ are the irrational numbers. Note that $4.2122122212222\ldots$ is irrational because it is a nonrepeating, nonterminating decimal.

(f) All of the numbers are real numbers.

Classroom Example 5

List the numbers in the set

$\left\{-3, \frac{8}{3}, 7, 2.757557555\ldots, -\frac{11}{6}, 2.5, \sqrt{3}, 13, \sqrt{23}, 0\right\}$

that are

(a) Natural numbers
(b) Whole numbers
(c) Integers
(d) Rational numbers
(e) Irrational numbers
(f) Real numbers

Answer:
(a) 7, 13
(b) 0, 7, 13
(c) $-3, 0, 7, 13$
(d) $-3, 0, 7, 13, \frac{8}{3}, -\frac{11}{6}, 2.5$
(e) $2.757557555\ldots, \sqrt{3}, \sqrt{23}$
(f) All the numbers are real numbers.

Quick ✓

15. The whole numbers are numbers in the set $\{0, 1, 2, 3, 4, \ldots\}$.

16. The numbers in the set $\left\{x \middle| x = \frac{p}{q}\text{, where } p \text{ and } q \text{ are integers, } q \neq 0\right\}$ are called rational numbers.

17. The set of irrational numbers have decimal representations that neither terminate nor repeat.

18. *True or False* The rational numbers are a subset of the set of irrational numbers. **18.** False

19. *True or False* If a number is expressed as a decimal, then it is rational. False

20. *True or False* Every rational number is a real number. True

21. *True or False* If a number is rational, it cannot be irrational. True

In Problems 22–27, list the numbers in the set $\left\{\frac{7}{3}, -9, 10, 4.\overline{56}, 5.7377377737777\ldots, \frac{0}{3}, \pi, -\frac{4}{7}, \frac{12}{4}, \sqrt{11}\right\}$ *that are*

22. Natural numbers $10, \frac{12}{4}$

23. Whole numbers $10, \frac{0}{3}, \frac{12}{4}$

24. Integers $-9, 10, \frac{0}{3}, \frac{12}{4}$

25. Rational numbers $\frac{7}{3}, -9, 10, 4.\overline{56}, \frac{0}{3}, -\frac{4}{7}, \frac{12}{4}$

26. Irrational numbers **26.** $5.7377377737777\ldots, \pi, \sqrt{11}$

27. Real numbers All

3 Approximate Decimals by Rounding or Truncating

Every number written in decimal form is a real number that may either be rational or irrational. In addition, every real number can be represented by a decimal. For example, the rational number $\frac{3}{4}$ can be written in decimal form as 0.75. The rational number $\frac{2}{3}$ is equivalent to $0.666\ldots$ or $0.\overline{6}$ as a decimal.

Irrational numbers have decimals that neither terminate nor repeat. The irrational numbers $\sqrt{2}$ and π have decimal representations that begin as follows:

$$\sqrt{2} = 1.414213\ldots \qquad \pi = 3.14159\ldots$$

Approximations are used to write irrational numbers as decimals. The symbol $\approx$ (which is read "approximately equal to") is used to do so. For example,

$$\sqrt{2} \approx 1.4142 \qquad \pi \approx 3.1416$$

In approximating decimals, either *round* or *truncate* to a given number of decimal places. The number of decimal places determines the location of the *final digit* in the decimal approximation.

Work Smart

When should you truncate and when should you round? If you have 7 pieces of pizza for four people, how many whole pieces should each person get? If you divide, you get 1.75 pieces. Rounding to the nearest whole digit gives 2 pieces and truncating gives 1 piece. Better truncate to make sure you have enough for everyone!

Definition

Truncation: Drop all the digits that follow the specified final digit in the decimal.
Rounding: Identify the specified final digit in the decimal. If the next digit is 5 or more, add 1 to the final digit; if the next digit is 4 or less, leave the final digit as it is. Then truncate all digits to the right of the final digit.

EXAMPLE 6

Approximating a Decimal by Truncating and by Rounding

Approximate 13.9463 to two decimal places by

(a) Truncating **(b)** Rounding

Classroom Example 6
Approximate 7.7291 to two decimal places by
(a) Truncating (b) Rounding
Answer: (a) 7.72 (b) 7.73

Solution

We want to approximate the decimal to two decimal places, so the final digit is 4:

$$13.9463$$

(a) To truncate, remove all digits to the right of the final digit, 4, to get 13.94.

(b) To round to two decimal places, find the digit to the right of the final digit. It is 6. Because 6 is 5 or more, add 1 to the final digit 4 to get 13.95.

Quick ✓

In Problems 28 and 29, approximate each number by (a) truncating and (b) rounding to the indicated number of decimal places.

28. 5.694392; three decimal places
(a) 5.694 **(b)** 5.694

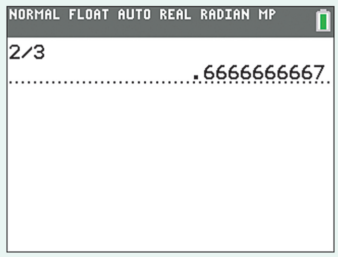
29. -4.9369102; two decimal places
(a) -4.93 **(b)** -4.94

The Graphing Calculator: Does Your Calculator Truncate or Round?

Calculators are capable of displaying only a certain number of decimal places. Most scientific calculators display eight digits, and most graphing calculators display ten to twelve digits. When a number has more digits than the calculator can display, the calculator will either round or truncate.

To see whether your calculator rounds or truncates, divide 2 by 3. How many digits do you see? Is the last digit a 6 or a 7? If it is a 6, your calculator truncates; if it is a 7, your calculator rounds. Figure 3 shows the result on a graphing calculator. Does the calculator shown in Figure 3 round or truncate? Because the last digit displayed is a 7, it rounds.

Figure 3



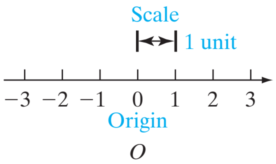
Teaching Tip
Remind students that the scale must consist of numbers that are evenly spaced.

4 Plot Points on the Real Number Line

The real numbers can be represented by points on a line called the **real number line.** Every real number corresponds to a point on the line, and each point on the line has a unique real number associated with it.

To construct a real number line, pick a point on a line somewhere in the center, and label it O. This point, called the **origin,** corresponds to the real number 0. See Figure 4. The point 1 unit to the right of O corresponds to the real number 1. The distance between 0 and 1 determines the **scale** of the number line. For example, the point associated with the number 2 is twice as far from O as 1 is. Notice that an arrowhead on the right end of the line indicates the direction in which the numbers increase. Points to the left of the origin correspond to the real numbers -1, -2, and so on.

Figure 4
The real number line.



In Other Words
Think of the real number line as the graph of the set of all real numbers.

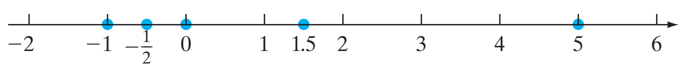
Definition
The real number associated with a point P is called the **coordinate** of P. The **real number line** is the set of all points that have been assigned coordinates.

EXAMPLE 7 **Plotting Points on the Real Number Line**

On the real number line, plot the points with coordinates 0, 5, -1, 1.5, $-\frac{1}{2}$.

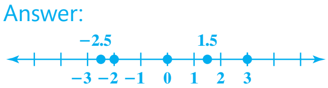
Solution
Draw a real number line with a scale of 1 and then plot the points. See Figure 5.

Figure 5



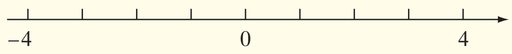
Classroom Example 7
On the real number line, plot the points with the coordinates 3, -2, 1.5, -2.5, and 0.

Answer:



Quick ✓

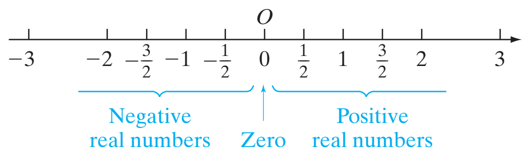
30. On the real number line, plot the points with coordinates 0, 3, -2, $\frac{1}{2}$, and 3.5.



See Graphing Answer Section.

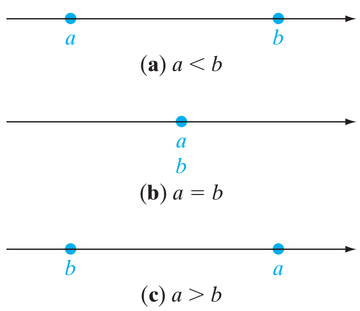
The real number line consists of three classes (or categories) of real numbers, as shown in Figure 6.

Figure 6



- The **negative real numbers** are the coordinates of points to the left of the origin O.
- The real number **zero** is the coordinate of the origin O.
- The **positive real numbers** are the coordinates of points to the right of the origin O.

Figure 7



5 Use Inequalities to Order Real Numbers

Given two numbers (points) a and b, a must be to the left of b, the same as b, or to the right of b. See Figure 7.

If a is to the left of b, say "a is less than b" and write $a < b$. If a is at the same location as b, then say that "a is equal to b" and write $a = b$. If a is to the right of b, say "a is greater than b" and write $a > b$. If a is either less than or equal to b, write $a \leq b$. Similarly, $a \geq b$ means that a is either greater than or equal to b. Collectively, the symbols $<$, $>$, $\leq$, and $\geq$ are called **inequality symbols.**

Note that $a < b$ and $b > a$ mean the same thing, so $2 < 3$ (2 is to the left of 3) may also be written $3 > 2$ (3 is to the right of 2).

EXAMPLE 8 Using Inequality Symbols

Classroom Example 8
Fill in the blank with the symbol $<$, $>$, or $=$ to make the sentence true.
(a) 7 __ 1 (b) -8 __ -3
(c) $\frac{2}{5}$ __ 0.4 (d) $\frac{3}{8}$ __ $\frac{1}{3}$

Answer: (a) $>$ (b) $<$
(c) $=$ (d) $>$

(a) $2 < 5$ because the coordinate 2 lies to the left of the coordinate 5 on the real number line.

(b) $-1 > -3$ because -1 lies to the right of -3 on the real number line.

(c) $3.5 \leq \frac{7}{2}$ because $3.5 = \frac{7}{2}$.

(d) $\frac{5}{6} > \frac{4}{5}$ because $\frac{5}{6} = 0.8\overline{3}$ and $\frac{4}{5} = 0.8$, so $\frac{5}{6}$ lies to the right of $\frac{4}{5}$ on the real number line.

Quick ✓

In Problems 31–36, replace the question mark by $<$, $>$, or $=$, whichever is correct.

31. $3 ? 6$ $<$ **32.** $-3 ? -2$ $<$ **33.** $\frac{2}{3} ? \frac{1}{2}$ $>$

34. $\frac{5}{7} ? 0.7$ $>$ **35.** $\frac{2}{3} ? \frac{10}{15}$ $=$ **36.** $\pi ? 3.14$ $>$

Note to Instructor
Quick Check exercises are designed as homework. For best results, assign them.

Work Smart: Study Skills
Selected problems in exercise sets are underlined. For extra help, you can find worked video solutions to these problems in MyMathLab or by using the QR code in each section.

Notice in Example 8 that the inequality symbol always points to the smaller number.

Inequalities of the form $a < b$ or $b > a$ are **strict inequalities,** whereas inequalities of the form $a \leq b$ or $b \geq a$ are **nonstrict** (or **weak**) **inequalities**.

Based upon the discussion so far, we conclude that

$$a > 0 \quad \text{is equivalent to} \quad a \text{ is positive}$$
$$a < 0 \quad \text{is equivalent to} \quad a \text{ is negative}$$

Read $a > 0$ as "a is positive" or "a is greater than 0." If $a \geq 0$, then either $a > 0$ or $a = 0$, and read this as "a is nonnegative" or "a is greater than or equal to 0."

R.1 Exercises

MyLabMath® Underlined exercise numbers have complete video solutions in MyLabMath or may be accessed using the QR code to the right.

Problems **1–36** *are the* **Quick ✓s** *that follow the* **EXAMPLES.**

Building Skills

In Problems 37–42, write each set using the roster method. See Objective 1.

37. $\{x \mid x \text{ is a whole number less than } 6\}$ $\{0, 1, 2, 3, 4, 5\}$

38. $\{x \mid x \text{ is a natural number less than } 4\}$ $\{1, 2, 3\}$

39. $\{x \mid x \text{ is an integer between } -3 \text{ and } 5\}$ $\{-2, -1, 0, 1, 2, 3, 4\}$

40. $\{x \mid x \text{ is an integer between } -4 \text{ and } 6\}$ $\{-3, -2, -1, 0, 1, 2, 3, 4, 5\}$

41. $\{x \mid x \text{ is a natural number less than } 1\}$ $\emptyset$ or $\{\ \}$

42. $\{x \mid x \text{ is a whole number less than } 0\}$ $\emptyset$ or $\{\ \}$

In Problems 43–50, let $A = \{1, 3, 5, 7, 9\}$, $B = \{2, 4, 6, 8\}$, $C = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, *and* $D = \{8, 6, 4, 2\}$. *Write True or False for each statement. Then justify your answer. See Objective 1.*

43. $B \subseteq C$ True
44. $A \subseteq C$ True
45. $B \subset D$ False
46. $A \subset C$ True
47. $B = D$ True
48. $B \subseteq D$ True
49. $\emptyset \subset C$ True
50. $\emptyset \subset B$ True

In Problems 51–54, fill in the blank with the appropriate symbol, $\in$ *or* $\notin$. *See Objective 1.*

51. $\frac{1}{2}$ __$\notin$__ $\{x \mid x \text{ is an integer}\}$

52. $4.\overline{5}$ __$\in$__ $\{x \mid x \text{ is a rational number}\}$

53. π __$\in$__ $\{x \mid x \text{ is a real number}\}$

54. 0 __$\notin$__ $\{x \mid x \text{ is a natural number}\}$

In Problems 55–58, list the numbers in each set that are (a) Natural numbers, (b) Integers, (c) Rational numbers, (d) Irrational numbers, (e) Real numbers. See Objective 2.

55. $A = \left\{-5, 4, \frac{4}{3}, -\frac{7}{5}, 5.\overline{1}, \pi\right\}$ (a) 4 (b) $-5, 4$

56. $B = \{13, 0, -4.5656\ldots, 2.43, \sqrt{2}\}$ (a) 13 (b) 0, 13

57. $C = \left\{100, -5.423, \frac{8}{7}, \sqrt{2} + 4, -64\right\}$ (a) 100 (b) $100, -64$

58. $D = \left\{15, -\frac{6}{1}, 7.3, \sqrt{2} + \pi\right\}$ (a) 15 (b) $-\frac{6}{1}, 15$

In Problems 59–62, approximate each number (a) by truncating and (b) by rounding to the indicated number of decimal places. See Objective 3.

59. 19.93483; 4 decimal places (a) 19.9348 (b) 19.9348

60. -93.432101; 2 decimal places (a) -93.43 (b) -93.43

61. 0.06345; 1 decimal place (a) 0.0 (b) 0.1

62. 9.9999; 2 decimal places (a) 9.99 (b) 10.00

In Problems 63 and 64, plot the points whose coordinates are given on the real number line. See Objective 4.

63. $2, 0, -3, 1.5, -\frac{3}{2}$

64. $4, -5, 2.5, \frac{5}{3}, -\frac{1}{2}$

63–64. See Graphing Answer Section.

In Problems 65–70, replace the question mark by <, >, or =, whichever is correct. See Objective 5.

65. $-5 \ ? \ -3$ $<$
66. $4 \ ? \ 2$ $>$
67. $\frac{3}{2} \ ? \ 1.5$ $=$
68. $\frac{2}{3} \ ? \ \frac{2}{5}$ $>$
69. $\frac{1}{3} \ ? \ 0.3$ $>$
70. $-\frac{8}{3} \ ? \ -\frac{8}{5}$ $<$

55. (c) $-5, 4, \frac{4}{3}, -\frac{7}{5}, 5.\overline{1}$ (d) π
55. (e) $-5, 4, \frac{4}{3}, -\frac{7}{5}, 5.\overline{1}, \pi$

56. (c) $-4.5656\ldots, 0, 2.43, 13$ (d) $\sqrt{2}$ (e) $-4.5656\ldots, 0, \sqrt{2}, 2.43, 13$

57. (c) $100, -5.423, \frac{8}{7}, -64$ (d) $\sqrt{2} + 4$ (e) $100, -5.423, \frac{8}{7}, \sqrt{2} + 4, -64$

58. (c) $-\frac{6}{1}, 7.3, 15$ (d) $\sqrt{2} + \pi$ (e) $-\frac{6}{1}, \sqrt{2} + \pi, 7.3, 15$

Applying the Concepts

71. Death Valley Death Valley in California is the lowest point in the United States, with an elevation that is 282 feet below sea level. Express this elevation as an integer. (SOURCE: *Information Please Almanac*) -282 feet

72. Dead Sea The Dead Sea, Israel–Jordan, is the lowest point in the world, with an elevation that is 1349 feet below sea level. Express this elevation as an integer. (SOURCE: *Information Please Almanac*) -1349 feet

73. Macy's Macy's lost \$4.39 per share of common stock in 2017. Express this loss as a rational number written in decimal form. (SOURCE: *Yahoo!Finance*) $-\$4.39$

74. Bitcoin Bitcoin stock lost \$1,859.76 in a recent trading day. Express this loss as a rational number written in decimal form. (SOURCE: *Yahoo!Finance*) $-\$1,859.76$

75. Golf In the game of golf, your score is often given in relation to par. For example, if par is 72 and a player shoots a 66, then he is 6 under par. Express this score as an integer. -6

76. It's a Little Chilly! The normal high temperature in Las Vegas, Nevada, on December 21 is 55°F. On December 21, 2017, the temperature was 5°F below normal. Express the departure from normal as an integer. (SOURCE: *USA Today*) -5°F

Extending the Concepts 77–80. Answers will vary.

77. Research the history of the set of irrational numbers. Your research should concentrate on the Greek cult called the Pythagoreans. Write a report on your findings.

78. Research the origins of the number 0. Is there any single person who can claim its discovery?

79. The first known computation of the decimal approximation of the number π is attributed to Archimedes around 200 B.C. Research Archimedes and find out his approximation.

80. The irrational number e is attributed to Leonhard Euler. Research Euler and find out the decimal approximation of e.

Explaining the Concepts 81–88. See Graphing Answer Section.

81. Are there any real numbers that are both rational and irrational? Are there any real numbers that are neither? Explain your reasoning. No; No

82. Explain why the sum of a rational number and an irrational number must be irrational.

83. Explain what a set is. Give an example of a set.

84. Explain why it is impossible to list the set of rational numbers using the roster method.

85. Explain the difference between a subset and a proper subset.

86. Describe the difference between 0.45 and $0.\overline{45}$. Are both rational? Yes

87. Explain the circumstances under which rounding and truncating will both result in the same decimal approximation.

88. Is there a positive real number "closest" to 0? No

Technology Exercises

89. Use technology to express $\frac{8}{7}$ rounded to three decimal places. Express $\frac{8}{7}$ truncated to three decimal places. 1.143; 1.142

90. Use technology to express $\frac{19}{7}$ rounded to four decimal places. Express $\frac{19}{7}$ truncated to four decimal places. 2.7143; 2.7142

R.2 Operations on Signed Numbers; Properties of Real Numbers

Objectives

1. Compute the Absolute Value of a Real Number
2. Add and Subtract Signed Numbers
3. Multiply and Divide Signed Numbers
4. State the Associative and Distributive Properties

The symbols used in algebra for the operations of addition, subtraction, multiplication, and division and the words used to describe the results of these operations are shown in Table 1 below.

Table 1

Operation	Symbol	Result of Operation
Addition	$a + b$	**Sum:** a plus b
Subtraction	$a - b$	**Difference:** a minus b b is subtracted from a
Multiplication	$a \cdot b$, $(a) \cdot b$, $a \cdot (b)$, $(a) \cdot (b)$, ab, $(a)b$, $a(b)$, $(a)(b)$	**Product:** a times b
Division	a/b or $\dfrac{a}{b}$	**Quotient:** a divided by b

In algebra, avoid using the multiplication sign $\times$ (to avoid confusion with the often-used x) and the division sign $\div$ (to avoid confusion with $+$). Also, two expressions placed next to each other without an operation symbol, as in ab or $(a)(b)$, are understood to be **factors** that are to be multiplied.

It is also preferable not to use mixed numbers in algebra. When mixed numbers are used, addition is understood—for example, $2\frac{3}{4}$ means $2 + \frac{3}{4}$. But in algebra, the absence of an operation symbol between two terms is taken to mean multiplication. To avoid any confusion, $2\frac{3}{4}$ will be written as 2.75 in decimal format or as an improper fraction, $\frac{11}{4}$.

The symbol $=$, which is called an **equal sign** and read as "equals" or "is," is used to express the idea that the expression on the left side of the equal sign is equivalent to the expression on the right.

1 Compute the Absolute Value of a Real Number

The real number line is used to describe the concept of *absolute value*.

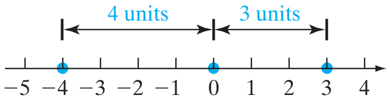
Work Smart

The absolute value of a number can never be negative because it represents a distance.

Definition

The **absolute value** of a number a, written $|a|$, is the distance from 0 to a on the real number line.

Figure 8



For example, because the distance from 0 to 3 on the real number line is 3, the absolute value of 3, $|3|$, is 3. Because the distance from 0 to -4 on the real number line is 4, $|-4| = 4$. See Figure 8.

EXAMPLE 1 Computing Absolute Value

(a) $|8| = 8$ because the distance from 0 to 8 on the real number line is 8.

(b) $|-5| = 5$ because the distance from 0 to -5 on the real number line is 5.

(c) $|0| = 0$ because the distance from 0 to 0 on the real number line is 0.

Classroom Example 1
Find the absolute value:
(a) $|3|$ (b) $|-9|$
Answer: (a) 3 (b) 9

Quick ✓

1. In the expression $a \cdot b$, the expressions a and b are called factors.
2. *True or False* The absolute value of a number is always positive.
   False because $|0| = 0$.

In Problems 3 and 4, evaluate each expression.

3. $|6|$ 6
4. $|-10|$ 10

2 Add and Subtract Signed Numbers

The following rules are used to add two real numbers.

In Other Words

To add two real numbers that have the same sign, add their absolute values and then keep the sign of the original numbers. To add two real numbers with different signs, subtract the smaller absolute value from the larger absolute value. The sign of the sum will be the sign of the number whose absolute value is larger.

Adding Two Nonzero Real Numbers

The approach to adding two real numbers depends on the signs of the two numbers.

1. *Both Positive:* Add the numbers.
2. *Both Negative:* Add the absolute values of the numbers. The sum will be negative.
3. *One Positive, One Negative:* Determine the absolute value of each number. Subtract the smaller absolute value from the larger absolute value.
   - If the larger absolute value was originally the positive number, then the sum is positive.
   - If the larger absolute value was originally the negative number, then the sum is negative.
   - If the two absolute values are equal, then the sum is 0.

EXAMPLE 2

Adding Two Real Numbers

Classroom Example 2
Perform the indicated operation.
(a) $-7 + (-6)$
(b) $-12 + 4$
(c) $8.1 + (-3.7)$

Answer: (a) -13 (b) -8 (c) 4.4

Perform the indicated operation.

(a) $-2 + (-3)$ **(b)** $9.3 + (-6.4)$

Solution

(a) Both numbers are negative, so first find their absolute values: $|-2| = 2$ and $|-3| = 3$. Now add the absolute values: $2 + 3 = 5$. Because both numbers to be added are negative, the sum must be negative. So

$$-2 + (-3) = -5$$

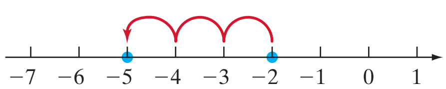
(b) One number is positive and the other is negative. Find the absolute value of each number: $|9.3| = 9.3$ and $|-6.4| = 6.4$. Subtract the smaller absolute value from the larger absolute value and get $9.3 - 6.4 = 2.9$. The larger absolute value was originally positive, so the sum is positive:

$$9.3 + (-6.4) = 2.9$$

Work Smart

Another way to add two real numbers is to use a number line. In Example 2(a), place a point at -2 on the real number line. Then move three places to the left (since -3 is being added). See Figure 9. End up at the point whose coordinate is -5, so $-2 + (-3) = -5$.

Figure 9



Quick ✓

In Problems 5–9, perform the indicated operation.

5. $18 + (-6)$ 12 **6.** $-21 + 10$ -11 **7.** $-5.4 + (-1.2)$ -6.6

8. $-6.5 + 4.3$ -2.2 **9.** $-9 + 9$ 0

What do you notice about the sum in Problem 9 from the Quick Check above? The result of Problem 9 is true in general.

Work Smart

The additive inverse of a, $-a$, is sometimes called the *negative* of a or the *opposite* of a. Be careful when using these terms because they suggest that the additive inverse is a negative number, which may not be true! For example, the additive inverse of -3 is 3, a positive number.

Teaching Tip

Remind students that "the opposite of a" might be a positive number.

Additive Inverse Property

For any real number a, there is a real number $-a$, called the **additive inverse,** or **opposite,** of a, having the following property:

$$a + (-a) = -a + a = 0$$

For example, the additive inverse or opposite of 3 is -3, so $-3 + 3 = 0$. The opposite of -42 is 42, and the opposite of $-\frac{5}{8}$ is $\frac{5}{8}$. This gives the following property:

Double Negative Property

For any real number a,

$$-(-a) = a$$

EXAMPLE 3 Finding an Additive Inverse

Classroom Example 3

State the additive inverse of

(a) -11 (b) 42

Answer:

(a) 11 (b) -42

(a) The additive inverse of 6 is -6 because $6 + (-6) = 0$.

(b) The additive inverse of -10 is $-(-10) = 10$ because $-10 + 10 = 0$.

Quick ✓

10. For any real number a, there is a real number $-a$, called the <u>additive</u> <u>inverse</u>, or <u>opposite</u>, of a such that $a + (-a) = -a + a = \underline{0}$.

11. For any real number a, $-(-a) = \underline{a}$.

In Problems 12–16, determine the additive inverse of the given real number.

12. 5 -5 **13.** $\frac{4}{5}$ $-\frac{4}{5}$ **14.** -12 12 **15.** $-\frac{5}{3}$ $\frac{5}{3}$ **16.** 0 0

The real number 0 is the only number that can be added to any real number a and result in the same number a. This is called the *Identity Property of Addition.*

Identity Property of Addition

For any real number a,

$$0 + a = a + 0 = a$$

That is, the sum of any number and 0 is that number. We call 0 the **additive identity**.

The Identity Property of Addition is used throughout the entire course (and future courses) to create a new expression that is equivalent to a previous expression. For example, $a = a + 0 = a + 6 + (-6)$ because $6 + (-6) = 0$.

Addition is also *commutative*. That is to say, the same result happens whether $a + b$ or $b + a$ is computed.

Commutative Property of Addition

If a and b are real numbers, then

$$a + b = b + a$$

Now a more formal definition of absolute value is presented.

In Other Words

The absolute value of a number greater than or equal to 0 is the number itself. The absolute value of a number less than zero is the additive inverse of the number.

Definition

The **absolute value** of a real number a, denoted by the symbol $|a|$, is defined by the rules

$$|a| = a \quad \text{if } a \geq 0 \qquad \text{and} \qquad |a| = -a \quad \text{if } a < 0$$

This definition of absolute value is sometimes called the algebraic definition. For example, $|12| = 12$ and $|-13| = -(-13) = 13$.

The additive inverse can be used to define subtraction between two real numbers.

In Other Words

To subtract b from a, add the "opposite" of b to a.

Definition

If a and b are real numbers, then the **difference** $a - b$, read "a minus b" or "a less b," is defined as

$$a - b = a + (-b)$$

Based on this definition, note that subtracting b from a is really just adding the additive inverse of b to a.

EXAMPLE 4 Working with Differences

Evaluate each expression.

(a) $10 - 4$ **(b)** $-7.3 - (-4.2)$

Classroom Example 4
Evaluate each expression.
(a) $13 - 9$ (b) $-8.6 - (-4.7)$
Answer: (a) 4 (b) -3.9

Solution

(a) $10 - 4 = 10 + (-4) = 6$

(b) Notice that a negative number is being subtracted. Subtracting -4.2 is the same as adding 4.2, so

$$\begin{aligned} -7.3 - (-4.2) &= -7.3 + 4.2 \\ &= -3.1 \end{aligned}$$

Quick ✓

In Problems 17–22, evaluate each expression.

17. $6 - 2$ 4 **18.** $4 - 13$ -9 **19.** $-3 - 8$ -11

20. $12.5 - 3.4$ 9.1 **21.** $-8.5 - (-3.4)$ -5.1 **22.** $-6.9 - 9.2$ -16.1

3 Multiply and Divide Signed Numbers

Multiplication is repeated addition. For example, $3 \cdot 5$ is equivalent to adding 5 three times. That is,

$$3 \cdot 5 = \underbrace{5 + 5 + 5}_{\text{Add 5 three times}} = 15$$

Also, because

$$5 \cdot 3 = 3 + 3 + 3 + 3 + 3 = 15$$

multiplication of two real numbers a and b is commutative, just like addition.

Commutative Property of Multiplication

If a and b are real numbers, then

$$a \cdot b = b \cdot a$$

When multiplying real numbers, follow these rules for determining the sign of the product:

Rules of Signs for Multiplying Two Real Numbers

1. The product of two positive real numbers is positive.
2. The product of one positive real number and one negative real number is negative.
3. The product of two negative real numbers is positive.

EXAMPLE 5 Multiplying Signed Numbers

(a) $3(-5) = -(3 \cdot 5) = -15$ **(b)** $-7 \cdot 3 = -(7 \cdot 3) = -21$

(c) $(-9)(-4) = 36$ **(d)** $-1.5(2.6) = -3.9$

Classroom Example 5

Find each product:

(a) $4(-2)$ (b) $-5 \cdot 6$

(c) $(-7)(-3)$ (d) $(-1.2)(3.4)$

Answer: (a) -8 (b) -30 (c) 21 (d) -4.08

Work Smart

In Example 5(a), $3(-5)$ means to add -5 three times. That is, $3(-5)$ means

$$-5 + (-5) + (-5)$$

which is -15. In Example 5(b),

$$\begin{aligned} -7 \cdot 3 &= 3 \cdot (-7) \\ &= -7 + (-7) + (-7) \\ &= -21 \end{aligned}$$

Do you see why a positive times a negative is negative?

Quick ✓

23. *True or False* The product of two negative real numbers is positive. True

24. *True or False* Addition and multiplication are commutative. True

In Problems 25–29, multiply.

25. $-6(8)$ -48 **26.** $12(-5)$ -60 **27.** $4 \cdot 14$ 56

28. $-7(-15)$ 105 **29.** $-1.9(-2.7)$ 5.13

The real number 1 has an interesting property. Recall that the expression $3 \cdot 5$ is equivalent to $5 + 5 + 5$. Therefore, $1 \cdot 5$ means to add 5 one time, so, $1 \cdot 5 = 5$. This result is true in general.

Identity Property of Multiplication

For any real number a,

$$a \cdot 1 = 1 \cdot a = a$$

1 is called the **multiplicative identity.**

In Other Words

The product of any number and 1 is that number.

The multiplicative identity lets us create equivalent expressions. For example, the expressions $\frac{4}{5}$ and $\frac{4}{5} \cdot \frac{3}{3}$ are equivalent because $\frac{3}{3} = 1$.

For each *nonzero* real number a, there is a real number $\frac{1}{a}$, called the *multiplicative inverse* of a, having the following property:

Multiplicative Inverse Property

For each *nonzero* real number a, there is a real number $\frac{1}{a}$, called the **multiplicative inverse** or **reciprocal** of a, having the following property:

$$a \cdot \frac{1}{a} = \frac{1}{a} \cdot a = 1 \qquad a \neq 0$$

In Other Words

When reciprocals are multiplied, their product is 1.

EXAMPLE 6 Finding the Multiplicative Inverse or Reciprocal

Classroom Example 6
Determine the multiplicative inverse of
(a) 7 (b) -2 (c) $-\frac{4}{5}$
Answer:
(a) $\frac{1}{7}$ (b) $-\frac{1}{2}$ (c) $-\frac{5}{4}$

(a) The multiplicative inverse, or reciprocal, of 5 is $\frac{1}{5}$.

(b) The multiplicative inverse, or reciprocal, of -4 is $-\frac{1}{4}$.

(c) The multiplicative inverse of $\frac{2}{3}$ is $\frac{3}{2}$ because $\frac{2}{3} \cdot \frac{3}{2} = 1$.

Work Smart
Do not confuse the multiplicative inverse or reciprocal with the additive inverse or opposite. The multiplicative inverse, or reciprocal, of a number does not change signs. It only reverses the numbers in the numerator and denominator.

Quick ✓

30. The additive inverse of a, $-a$, is also called the opposite of a. The multiplicative inverse of a, $\frac{1}{a}$, is also called the reciprocal of a.

In Problems 31–34, find the multiplicative inverse or reciprocal of the given real number.

31. 10 $\frac{1}{10}$ **32.** -8 $-\frac{1}{8}$ **33.** $\frac{2}{5}$ $\frac{5}{2}$ **34.** $-\frac{1}{5}$ -5

The idea behind the multiplicative inverse is used to define division of real numbers.

Definition

If a is a real number and b is a nonzero real number, the **quotient** $\frac{a}{b}$, read as "a divided by b" or "the ratio of a to b," is defined as

$$\frac{a}{b} = a \cdot \frac{1}{b} \qquad \text{if } b \neq 0$$

For example, $\frac{5}{8} = 5 \cdot \frac{1}{8}$. Because division of real numbers can be represented as multiplication, the same rules of signs that apply to multiplication also apply to division.

Rules of Signs for Dividing Two Real Numbers

1. The quotient of two positive numbers is positive.
2. The quotient of one positive real number and one negative real number is negative.
3. The quotient of two negative real numbers is positive.

Teaching Tip
Emphasize the placement of the negative sign in a numerical example to explain the property:

$$\frac{-3}{4} = \frac{3}{-4} = -\frac{3}{4}$$

Put another way, if a and b are real numbers and $b \neq 0$, then

$$-\frac{a}{b} = \frac{-a}{b} = \frac{a}{-b} \qquad \text{and} \qquad \frac{-a}{-b} = \frac{a}{b}$$

Additional properties of the numbers 0 and 1 are now introduced on the next page.

Multiplication by Zero

For any real number a, the product of a and 0 is always 0; that is,

$$a \cdot 0 = 0 \cdot a = 0$$

Teaching Tip

Division by 0 is not defined. One reason is to avoid the following problem: If $\frac{2}{0} = x$, then it must be the case that $2 = 0 \cdot x$. But according to the Multiplication by Zero Property, $0 \cdot x = 0$ for any real number x, so there is no number x such that $\frac{2}{0} = x$.

Division Properties

For any nonzero real number a,

$$\frac{0}{a} = 0 \qquad \frac{a}{a} = 1 \qquad \frac{a}{1} = a \qquad \frac{a}{0} \text{ is undefined}$$

Perhaps you are wondering what $\frac{0}{0}$ equals. The answer, which may surprise you, is that the value of $\frac{0}{0}$ cannot be determined. Why? If $\frac{0}{0} = n$, then $0 \cdot n = 0$. But $0 \cdot n = 0$ for any real number n. For this reason, we say that $\frac{0}{0}$ is **indeterminate.**

4 State the Associative and Distributive Properties

The order in which three real numbers are added or multiplied does not affect the final result. This property is called the *Associative Property*.

EXAMPLE 7 Illustrating the Associative Property

Classroom Example 7

Use Example 7.

(a) $17 + (3 + 5) = 17 + 8 = 25$
$(17 + 3) + 5 = 20 + 5 = 25$

Therefore,
$17 + (3 + 5) = (17 + 3) + 5$

(b) $13(2 \cdot 5) = 13 \cdot 10 = 130$
$(13 \cdot 2) \cdot 5 = 26 \cdot 5 = 130$

Therefore,
$13(2 \cdot 5) = (13 \cdot 2) \cdot 5$

Teaching Tip

Students may have been taught to use the Commutative and Associative Properties of Addition to regroup when adding columns of numbers to find numbers that sum to multiples of 10:

$$\begin{array}{rr} 19 & \left.\begin{array}{r}19\\21\end{array}\right\}40 \\ 34 & \\ 21 & \left.\begin{array}{r}34\\+46\end{array}\right\}80 \\ +46 & \\ \hline 120 & 120 \end{array}$$

Associative Properties of Addition and Multiplication

If a, b, and c are real numbers, then

$$a + (b + c) = (a + b) + c = a + b + c$$
$$a(bc) = (ab) \cdot c = abc$$

The next property will be used throughout the course.

Teaching Tip

The long name for the Distributive Property is the *Distributive Property of Multiplication over Addition.* Students may want to apply the Distributive Property to exponents later, so establishing this fact early is important.

The Distributive Property

If a, b, and c are real numbers, then

$$a(b + c) = ab + ac$$
$$(a + b) \cdot c = ac + bc$$

EXAMPLE 8

Using the Distributive Property

Classroom Example 8
Use the Distributive Property to remove the parentheses.
(a) $3(z + 4)$ (b) $-5(3p + 2)$
(c) $(n - 3) \cdot 2$

Answer: (a) $3z + 12$
(b) $-15p - 10$ (c) $2n - 6$

Use the Distributive Property to remove the parentheses.

(a) $2(x + 3)$ **(b)** $-3(2y + 1)$ **(c)** $(z - 4) \cdot 3$

Solution

(a) $2(x + 3) = 2 \cdot x + 2 \cdot 3 = 2x + 6$

(b) $-3(2y + 1) = -3 \cdot 2y + (-3)1 = -6y - 3$

(c) $(z - 4) \cdot 3 = z \cdot 3 - 4 \cdot 3 = 3z - 12$

When parentheses are preceded by a minus sign, such as in $-(3x + 7)$, the Distributive Property can be used as shown below.

$$-(3x + 7) = -1(3x + 7) = -1 \cdot 3x + -1 \cdot 7 = -3x - 7$$

We can generalize this property.

In Other Words
The opposite of a sum is the sum of the opposites.

The Opposite of a Sum

For any real numbers a and b,

$$-(a + b) = -a + (-b) = -a - b$$

Quick ✓

35. The Distributive Property states that $a(b + c) =$ __ab__ + __ac__.

36. $-(a + b) =$ __$-a - b$__

In Problems 37–42, use the Distributive Property to remove the parentheses.

37. $5(x + 3)$ $5x + 15$

38. $-6(x + 1)$ $-6x - 6$

39. $-4(z - 8)$ $-4z + 32$

40. $3(6x + 9)$ $18x + 27$

41. $-(11p + 8)$ $-11p - 8$

42. $-(-9 - 4t)$ $9 + 4t$

Summary Properties of the Real Number System

If a, b, and c are real numbers,

- Identity Properties $a + 0 = 0 + a = a$; $1 \cdot a = a \cdot 1 = a$
- Inverse Properties $a + (-a) = (-a) + a = 0$; $a \cdot \frac{1}{a} = \frac{1}{a} \cdot a = 1 \ (a \neq 0)$
- Double Negative Property $-(-a) = a$
- Commutative Properties $a + b = b + a$; $a \cdot b = b \cdot a$
- Multiplication Property of 0 $a \cdot 0 = 0 \cdot a = 0$
- Division Properties $\frac{0}{a} = 0, a \neq 0$; $\frac{a}{a} = 1, a \neq 0$; $\frac{a}{0}$ is undefined, $a \neq 0$
- Associative Properties $(a + b) + c = a + (b + c)$; $(ab)c = a(bc)$
- Distributive Property $a(b + c) = ab + ac$
- Opposite of a Sum $-(a + b) = -a - b$

Note to Instructor
Quick Check exercises are designed as homework. For best results, assign them.

R.2 Exercises

MyLabMath® Underlined exercise numbers have complete video solutions in MyLabMath or may be accessed using the QR code to the right.

Problems **1–42** *are the* **Quick ✓s** *that follow the* **EXAMPLES.**

Building Skills

In Problems 43–46, evaluate each expression. See Objective 1.

43. $\left|\frac{2}{3}\right|$ $\frac{2}{3}$ **44.** $\left|\frac{5}{6}\right|$ $\frac{5}{6}$ **45.** $\left|-\frac{8}{3}\right|$ $\frac{8}{3}$ **46.** $\left|-\frac{7}{2}\right|$ $\frac{7}{2}$

In Problems 47–52, perform the indicated operation. See Objective 2.

47. $-13 + 4$ -9

48. $-6 + 10$ 4

49. $12 - 5$ 7

50. $9 + (-3)$ 6

51. $4.3 - 6.8$ -2.5

52. $-8.2 - 4.5$ -12.7

In Problems 53–58, perform the indicated operation. See Objective 3.

53. $4(-8)$ -32

54. $-5(-15)$ 75

55. $-6(-14)$ 84

56. $7(-15)$ -105

57. $-4.3(8.5)$ -36.55

58. $-10.4(-0.6)$ 6.24

In Problems 59–66, use the Distributive Property to remove the parentheses. See Objective 4.

59. $2(x + 4)$ $2x + 8$

60. $3(y - 5)$ $3y - 15$

61. $-3(z - 2)$ $-3z + 6$

62. $-5(x - 4)$ $-5x + 20$

63. $(x - 10)\cdot 3$ $3x - 30$

64. $(3x + y)\cdot 2$ $6x + 2y$

65. $-(5z + 17)$ $-5z - 17$

66. $-(11 - 8k)$ $-11 + 8k$

Mixed Practice

In Problems 67–76, perform the indicated operation. Express all rational numbers in lowest terms.

67. $|13 - 16|$ 3

68. $|-4| + 12$ 16

69. $|6.2 - 9.5|$ 3.3

70. $|-5.4 + 10.5|$ 5.1

71. $-|-8(4)|$ -32

72. $-|-5\cdot 9|$ -45

73. $\frac{18}{0}$ undefined

74. $-\frac{7}{0}$ undefined

75. $\frac{0}{20}$ 0

76. $\frac{0}{5}$ 0

In Problems 77–82, state the property that is being illustrated.

77. $5\cdot 3 = 3\cdot 5$ Commutative Property of Multiplication

78. $9 + (-9) = 0$ Additive Inverse Property

79. $5\cdot\frac{1}{5} = 1$ Multiplicative Inverse Property

80. $3(x - 4) = 3x - 12$ Distributive Property

81. $3 + (4 + 5) = (3 + 4) + 5$ Associative Property of Addition

82. $\frac{0}{6} = 0$ Division Property

Applying the Concepts

83. Age of Presidents The youngest president at the time of inauguration was Theodore Roosevelt (42 years of age). The oldest president at the time of inauguration was Ronald Reagan (69 years of age). What is the difference in age at the time of inauguration between the oldest and youngest presidents? 27 years

84. Life Expectancy In South Korea, the life expectancy for females born in 1950 was 49 years of age. The life expectancy for females born in 2030 is expected to be 91 years of age. Compute the difference in life expectancy between 1950 and 2030. 42 years

85. Football The Chicago Bears obtained the following yardages for each of the first three plays of a game: 4, −3, 8. How many total yards did they gain for the first three plays? If 10 yards are required for a first down, did the Bears obtain a first down? 9 yards; No

86. Balancing a Checkbook At the beginning of the month, Paul had $400 in his checking account. During the month he wrote four checks for $20, $45, $60, and $105. He also made a deposit in the amount of $150. What is Paul's balance at the end of the month? $320

87. Peaks and Valleys In the United States, the highest elevation is Mount Denali in Alaska (20,320 feet above sea level); the lowest elevation is Death Valley in California (282 feet below sea level). What is the difference between the highest and lowest elevations? 20,602 feet

88. More Peaks and Valleys In Louisiana, the highest elevation is Driskill Mountain (535 feet above sea level); the lowest elevation is New Orleans (8 feet below sea level). What is the difference between the highest and lowest elevations? 543 feet

89–96. See Graphing Answer Section.

89. Illustrate why the product of two positive numbers is positive. Illustrate why the product of a positive number and a negative number is negative.

90. The Fibonacci Sequence The Fibonacci sequence is a famous sequence of numbers that were discovered by Leonardo Fibonacci of Pisa. The numbers in the sequence are 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, . . . , where each term after the second term is the sum of the two preceding terms.

(a) Compute the ratio of consecutive terms in the sequence. That is, compute $\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}$, and so on.
1; 2; 1.5; 1.66667; 1.6; 1.625; 1.615385; ...

(b) What number does the ratio approach? This number is called the **golden ratio** and has application in many different areas. 1.618

(c) Research Fibonacci numbers and cite three different applications.

Problems 91–94 use the following definition.

If P and Q are two points on a real number line with coordinates a and b, respectively, the **distance between P and Q,** denoted by $d(P, Q)$, is

$$d(P, Q) = |b - a|$$

Since $|b - a| = |a - b|$, it follows that $d(P, Q) = d(Q, P)$.

91. Plot the points $P = -4$ and $Q = 10$ on the real number line, and then find $d(P, Q)$. 14

92. Plot the points $P = -2$ and $Q = 6$ on the real number line, and then find $d(P, Q)$. 8

93. Plot the points $P = -3.2$ and $Q = 7.2$ on the real number line, and then find $d(P, Q)$. 10.4

94. Plot the points $P = -9.3$ and $Q = 1.6$ on the real number line, and then find $d(P, Q)$. 10.9

95. Why Is the Product of Two Negatives Positive? In this problem, we use the Distributive Property to illustrate why the product of two negative real numbers is positive.

(a) Express the product of any real number a and 0.

(b) Use the Additive Inverse Property to write 0 from part (a) as $b + (-b)$.

(c) Use the Distributive Property to distribute the a into the expression in part (b).

(d) Suppose that $a < 0$ and $b > 0$. What can be said about the product ab? Now, what must be true regarding the product $a(-b)$ in order for the sum to be zero?

96. Math for the Future Find the set of all ratios $\frac{p}{q}$ such that $p \in A$ and $q \in B$.

$$A = \{-9, -3, -1, 1, 3, 9\}, B = \{-4, -2, -1, 1, 2, 4\}$$

Explaining the Concepts

97–101. See Graphing Answer Section.

97. Explain why 0 does not have a multiplicative inverse.

98. Why does $2(4 \cdot 5)$ not equal $(2 \cdot 4) \cdot (2 \cdot 5)$?

99. Is subtraction associative? Support your conclusion with an example.

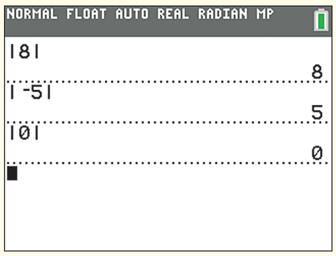
100. Is division commutative? Support your conclusion with an example.

101. Is division associative? Support your conclusion with an example.

Technology Exercises

Many calculators have the ability to compute absolute value. Figure 10 shows the results of Example 1 using a graphing calculator.

Figure 10



In Problems 102–107, use a calculator to perform the indicated operation. Express all rational numbers in lowest terms.

102. $-2.9 + (-6.3)$ -9.2

103. $5.4 - 9.2$ -3.8

104. $-5.4(-4.8)$ 25.92

105. $-3(6.4)$ -19.2

106. $|-3.65|(5.4)$ 19.71

107. $|4.5(-3.2)|$ 14.4

R.3 Perform Operations on Rational Numbers Written as Fractions

Objectives

1. Write Rational Numbers Written as Fractions in Lowest Terms
2. Multiply and Divide Rational Numbers Written as Fractions
3. Add and Subtract Rational Numbers Written as Fractions

In this section we present a review of operations on rational numbers written as fractions. Pay attention to the methods used throughout the section because these same methods apply to algebraic expressions written in fractional form.

1 Write Rational Numbers Written as Fractions in Lowest Terms

It is preferred to write rational numbers written as fractions in **lowest terms**—that is, without any common factors in the numerator and the denominator. The *Reduction Property* is used to obtain rational numbers in lowest terms.

Reduction Property

If a, b, and c are real numbers, then

$$\frac{ac}{bc} = \frac{a}{b} \qquad \text{if } b \neq 0, c \neq 0$$

EXAMPLE 1 Writing Rational Numbers Written as Fractions in Lowest Terms

Classroom Example 1

Write $\frac{18}{30}$ in lowest terms.

Answer: $\frac{3}{5}$

Divide out the common factor

$$\frac{45}{18} = \frac{\cancel{9} \cdot 5}{\cancel{9} \cdot 2} = \frac{5}{2}$$

Quick ✓

1. When a rational number is written so there are no common factors in the numerator and the denominator, the rational number is in lowest terms.

In Problems 2 and 3, write each rational number in lowest terms.

2. $\frac{13 \cdot 5}{13 \cdot 6}$ $\frac{5}{6}$

3. $\frac{80}{12}$ $\frac{20}{3}$

2 Multiply and Divide Rational Numbers Written as Fractions

Teaching Tip

Remind students that when multiplying and dividing rational numbers, do not find the LCD. Find the LCD when adding and subtracting.

We now review the methods for multiplying and dividing rational numbers.

Multiplying Rational Numbers

Step 1: Completely factor each integer in the numerator and denominator.

Step 2: Use the fact that if $\frac{a}{b}$ and $\frac{c}{d}$, $b \neq 0$, $d \neq 0$, are two rational numbers, then $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$.

Step 3: Divide out common factors in the numerator and denominator.

In Other Words

To multiply two rational numbers, multiply the numerators and then multiply the denominators. To divide two rational numbers, multiply the rational number in the numerator by the reciprocal of the rational number in the denominator.

Dividing Rational Numbers

To divide rational numbers, use the fact that if $\frac{a}{b}$ and $\frac{c}{d}$, $b \neq 0$, $c \neq 0$, $d \neq 0$, are two rational numbers, then $\dfrac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$.

EXAMPLE 2 Multiplying and Dividing Rational Numbers Written as Fractions

Classroom Example 2
Perform the indicated operation. Express the answer in lowest terms.

(a) $\frac{5}{7} \cdot \frac{56}{15}$ (b) $\dfrac{\frac{3}{10}}{\frac{12}{25}}$

Answer: (a) $\frac{8}{3}$ (b) $\frac{5}{8}$

Work Smart

In Example 2(a), rather than factoring each numerator and denominator individually and then dividing out like factors, you could proceed as follows:

$$\frac{10}{3} \cdot \frac{18}{25} = \frac{\overset{2}{\cancel{10}} \cdot \overset{6}{\cancel{18}}}{\underset{1}{\cancel{3}} \cdot \underset{5}{\cancel{25}}}$$

$$= \frac{2 \cdot 6}{1 \cdot 5}$$

Multiply numerators; multiply denominators: $= \frac{12}{5}$

Perform the indicated operation. Be sure to express the result in lowest terms.

(a) $\frac{10}{3} \cdot \frac{18}{25}$ **(b)** $\dfrac{\frac{14}{5}}{\frac{21}{10}}$

Solution

(a)

$$\frac{10}{3} \cdot \frac{18}{25} = \frac{2 \cdot 5}{3} \cdot \frac{9 \cdot 2}{5 \cdot 5}$$

$$= \frac{2 \cdot 5}{3} \cdot \frac{3 \cdot 3 \cdot 2}{5 \cdot 5}$$

Multiply; use the Reduction Property to divide out common factors: $= \frac{2 \cdot \cancel{5} \cdot \cancel{3} \cdot 3 \cdot 2}{\cancel{3} \cdot \cancel{5} \cdot 5}$

$$= \frac{2 \cdot 3 \cdot 2}{5}$$

Multiply: $= \frac{12}{5}$

(b) Rewrite the division problem as a multiplication problem by multiplying the numerator, $\frac{14}{5}$, by the reciprocal of the denominator, $\frac{21}{10}$. The reciprocal of $\frac{21}{10}$ is $\frac{10}{21}$.

$$\dfrac{\frac{14}{5}}{\frac{21}{10}} = \frac{14}{5} \cdot \frac{10}{21}$$

Factor: $= \frac{7 \cdot 2}{5} \cdot \frac{5 \cdot 2}{7 \cdot 3}$

Multiply; divide out common factors: $= \frac{\cancel{7} \cdot 2 \cdot \cancel{5} \cdot 2}{\cancel{5} \cdot \cancel{7} \cdot 3}$

$$= \frac{2 \cdot 2}{3} = \frac{4}{3}$$

Quick ✓

In Problems 4–7, perform the indicated operation. Express your answer in lowest terms.

4. $\frac{5}{7}\left(-\frac{21}{10}\right)$ $-\frac{3}{2}$ **5.** $\frac{35}{15} \cdot \frac{3}{14}$ $\frac{1}{2}$ **6.** $\dfrac{\frac{4}{5}}{\frac{12}{25}}$ $\frac{5}{3}$ **7.** $\frac{24}{35} \div \left(-\frac{8}{7}\right)$ $-\frac{3}{5}$

3 Add and Subtract Rational Numbers Written as Fractions

We now review addition and subtraction of rational numbers.

In Other Words
To add two rational numbers with a common denominator, add the numerators and write the sum over the common denominator.

Adding or Subtracting Rational Numbers

Step 1: If $\frac{a}{c}$ and $\frac{b}{c}$, $c \neq 0$, are two rational numbers, then $\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$ and $\frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}$.

Step 2: Write the result in lowest terms.

EXAMPLE 3

Adding or Subtracting Rational Numbers with Common Denominators

Classroom Example 3
Perform the indicated operation. Express the answer in lowest terms.
(a) $\frac{12}{35} + \frac{3}{35}$ (b) $\frac{3}{16} - \frac{9}{16}$

Answer: (a) $\frac{3}{7}$ (b) $-\frac{3}{8}$

Perform the indicated operation. Be sure to express the result in lowest terms.

(a) $\frac{7}{24} + \frac{11}{24}$ **(b)** $\frac{2}{15} - \frac{7}{15}$

Solution

(a)

$$\frac{7}{24} + \frac{11}{24} = \frac{7+11}{24}$$
$$= \frac{18}{24}$$
Factor numerator and denominator: $= \frac{6 \cdot 3}{6 \cdot 4}$
Divide out common factors: $= \frac{3}{4}$

(b)

$$\frac{2}{15} - \frac{7}{15} = \frac{2-7}{15}$$
$$= \frac{-5}{15}$$
Factor numerator and denominator: $= \frac{-1 \cdot 5}{3 \cdot 5}$
Divide out common factors: $= \frac{-1}{3} = -\frac{1}{3}$

Quick ✓

In Problems 8 and 9, perform the indicated operation. Express your answer in lowest terms.

8. $\frac{11}{12} + \frac{5}{12}$ $\frac{4}{3}$ **9.** $\frac{3}{18} - \frac{13}{18}$ $-\frac{5}{9}$

Classroom Example 4
Find the LCD of the rational numbers $\frac{5}{6}$ and $\frac{7}{15}$. Then rewrite each rational number with the LCD.
Answer: The LCD is 30; $\frac{5}{6} = \frac{25}{30}$ and $\frac{7}{15} = \frac{14}{30}$.

What if the denominators of the rational numbers to be added or subtracted are not the same? In this case, rewrite each rational number over a *least common denominator*. The **least common denominator (LCD)** is the smallest integer that is a multiple of each denominator.

EXAMPLE 4

How to Find the Least Common Denominator

Find the least common denominator of the rational numbers $\frac{7}{30}$ and $\frac{5}{12}$. Then rewrite each rational number with the least common denominator.

Step-by-Step Solution

Step 1: Factor each denominator as a product of prime factors, arranging like factors vertically.

$$30 = 5 \cdot 6 = 5 \cdot 3 \cdot 2$$
$$12 = 3 \cdot 4 = \quad\; 3 \cdot 2 \cdot 2$$

Work Smart
Line up the factors vertically to find the LCD.

Step 2: Find the product of each of the prime factors the greatest number of times they appear in any factorization.

$$\text{LCD} = 5 \cdot 3 \cdot 2 \cdot 2$$
$$= 60$$

Teaching Tip
Remind students of the definition of a prime number.

(*continued*)

The Multiplicative Identity Property, $1 \cdot a = a$, is used to write $\frac{7}{30}$ and $\frac{5}{12}$ with the denominator 60. Multiply $\frac{7}{30}$ by $1 = \frac{2}{2}$ and multiply $\frac{5}{12}$ by $1 = \frac{5}{5}$:

$$\frac{7}{30} = \frac{7}{30} \cdot \frac{2}{2} = \frac{7 \cdot 2}{30 \cdot 2} = \frac{14}{60} \qquad \frac{5}{12} = \frac{5}{12} \cdot \frac{5}{5} = \frac{5 \cdot 5}{12 \cdot 5} = \frac{25}{60}$$

Quick ✓

In Problems 10 and 11, find the least common denominator (LCD) of each pair of rational numbers. Then rewrite each rational number with the LCD.

10. $\frac{3}{25}$ and $\frac{2}{15}$ LCD $= 75$; $\frac{3}{25} = \frac{9}{75}$; $\frac{2}{15} = \frac{10}{75}$

11. $\frac{5}{18}$ and $-\frac{1}{63}$ LCD $= 126$; $\frac{5}{18} = \frac{35}{126}$; $-\frac{1}{63} = -\frac{2}{126}$

Classroom Example 5
Perform the indicated operation:
(a) $\frac{8}{15} - \frac{7}{10}$ (b) $\frac{7}{2} + \frac{5}{3}$

Answer: (a) $-\frac{1}{6}$ (b) $\frac{31}{6}$

EXAMPLE 5 How to Add or Subtract Fractions Using the Least Common Denominator

Perform the indicated operation:

(a) $\frac{5}{2} + \frac{4}{3}$ **(b)** $\frac{5}{28} - \frac{5}{12}$

Step-by-Step Solution

(a) $\frac{5}{2} + \frac{4}{3}$

Step 1: Find the least common denominator.

Each denominator is prime, so the LCD $= 2 \cdot 3 = 6$.

Step 2: Rewrite each rational number with the common denominator.

$$\frac{5}{2} + \frac{4}{3} = \frac{5}{2} \cdot \frac{3}{3} + \frac{4}{3} \cdot \frac{2}{2} = \frac{15}{6} + \frac{8}{6}$$

Step 3: Add the numerators and write the result over the common denominator.

$$= \frac{15 + 8}{6} = \frac{23}{6}$$

Step 4: Write the result in lowest terms.

The rational number is already in lowest terms.
So, $\frac{5}{2} + \frac{4}{3} = \frac{23}{6}$

(b) $\frac{5}{28} - \frac{5}{12}$

Step 1: Find the least common denominator.

$$\begin{aligned} 28 &= 7 \cdot 4 &&= 7 \cdot 2 \cdot 2 \\ 12 &= \phantom{7 \cdot{}} 4 \cdot 3 &&= \phantom{7 \cdot{}} 2 \cdot 2 \cdot 3 \\ \text{LCD} &&&= 7 \cdot 2 \cdot 2 \cdot 3 \\ &&&= 84 \end{aligned}$$

Step 2: Rewrite each rational number with the common denominator.

$$\frac{5}{28} - \frac{5}{12} = \frac{5}{28} \cdot \frac{3}{3} - \frac{5}{12} \cdot \frac{7}{7}$$
$$= \frac{15}{84} - \frac{35}{84}$$

Step 3: Subtract the numerators and write the result over the common denominator.

$$= \frac{15 - 35}{84}$$
$$= \frac{-20}{84}$$

Step 4: Write the result in lowest terms.

$$= \frac{-5 \cdot 4}{21 \cdot 4}$$
$$= \frac{-5}{21}$$
$$= -\frac{5}{21}$$

The steps used in Example 5 are summarized below.

Adding or Subtracting Rational Numbers with Unlike Denominators

Step 1: Find the least common denominator.

Step 2: Rewrite each rational number with the common denominator.

Step 3: Add or subtract the numerators, and write the result over the common denominator.

Step 4: Write the result in lowest terms.

Quick ✓

In Problems 12–14, perform the indicated operation. Express your answer in lowest terms.

12. $\frac{3}{4} + \frac{1}{5}$ $\frac{19}{20}$

13. $\frac{3}{20} + \frac{2}{15}$ $\frac{17}{60}$

14. $\frac{5}{14} - \frac{11}{21}$ $-\frac{1}{6}$

R.3 Exercises

MyLabMath® Underlined exercise numbers have complete video solutions in MyLabMath or may be accessed using the QR code to the right.

Problems **1–14** *are the* **Quick✓s** *that follow the* **EXAMPLES**.

Building Skills

In Problems 15–20, write each rational number in lowest terms. See Objective 1.

15. $\frac{4}{12}$ $\frac{1}{3}$

16. $\frac{6}{18}$ $\frac{1}{3}$

17. $-\frac{15}{35}$ $-\frac{3}{7}$

18. $-\frac{12}{28}$ $-\frac{3}{7}$

19. $\frac{-50}{-10}$ 5

20. $\frac{81}{-27}$ -3

In Problems 21–32, multiply or divide the rational numbers. Express each product or quotient as a rational number in lowest terms. See Objective 2.

21. $\frac{3}{4}\cdot\frac{20}{9}$ $\frac{5}{3}$

22. $\frac{2}{3}\cdot\frac{15}{6}$ $\frac{5}{3}$

23. $-\frac{5}{6}\cdot\frac{18}{5}$ -3

24. $-\frac{9}{8}\cdot\frac{16}{3}$ -6

25. $\frac{5}{8}\cdot\frac{2}{15}$ $\frac{1}{12}$

26. $\frac{3}{14}\cdot\frac{7}{12}$ $\frac{1}{8}$

27. $\frac{5}{2}\div\frac{25}{4}$ $\frac{2}{5}$

28. $\frac{2}{3}\div\frac{8}{9}$ $\frac{3}{4}$

29. $\dfrac{-\frac{6}{5}}{\frac{8}{15}}$ $-\frac{9}{4}$

30. $\dfrac{\frac{12}{7}}{-\frac{18}{21}}$ -2

31. $\dfrac{-\frac{9}{2}}{-\frac{3}{4}}$ 6

32. $\dfrac{-\frac{10}{7}}{-\frac{5}{14}}$ 4

In Problems 33–51, add or subtract the rational numbers. Express each sum or difference as a rational number in lowest terms. See Objective 3.

33. $\frac{5}{3}+\frac{1}{3}$ 2

34. $\frac{3}{4}+\frac{9}{4}$ 3

35. $\frac{11}{6}-\frac{1}{6}$ $\frac{5}{3}$

36. $\frac{19}{6}-\frac{5}{6}$ $\frac{7}{3}$

37. $-\frac{3}{4}+\frac{1}{3}$ $-\frac{5}{12}$

38. $\frac{-1}{7}+\frac{4}{9}$ $\frac{19}{63}$

39. $\frac{1}{4}+\frac{5}{6}$ $\frac{13}{12}$

40. $\frac{5}{8}+\frac{5}{12}$ $\frac{25}{24}$

41. $-\frac{3}{10}-\frac{7}{15}$ $-\frac{23}{30}$

42. $\frac{7}{16}-\frac{9}{20}$ $-\frac{1}{80}$

43. $\frac{5}{18}-\frac{11}{15}$ $-\frac{41}{90}$

44. $\frac{5}{24}+\frac{7}{32}$ $\frac{41}{96}$

45. $-\frac{7}{8}+\frac{3}{10}$ $-\frac{23}{40}$

46. $-\frac{7}{12}+\frac{2}{15}$ $-\frac{9}{20}$

47. $\frac{7}{24}-\frac{3}{20}$ $\frac{17}{120}$

48. $\frac{3}{28}+\frac{5}{12}$ $\frac{11}{21}$

49. $-\frac{7}{9}-\frac{2}{15}$ $-\frac{41}{45}$

50. $-\frac{5}{18}-\frac{1}{45}$ $-\frac{3}{10}$

51. $\frac{-4}{25}-\frac{7}{30}$ $-\frac{59}{150}$

Explaining the Concepts

52–54. See Graphing Answer Section.

52. Explain how to write a rational number in lowest terms.

53. Explain how to find the least common denominator of two rational numbers.

54. Explain how to add two rational numbers that do not have a common denominator.

R.4 Exponents; Order of Operations

Objectives

1. Evaluate Real Numbers with Exponents
2. Use the Order of Operations to Evaluate Expressions

1 Evaluate Real Numbers with Exponents

Integer exponents indicate repeated multiplication. For example,

$$3^4 = 3 \cdot 3 \cdot 3 \cdot 3 = 81$$

In the expression 3^4, 3 is the **base** and 4 is the **exponent** or **power.** The power tells us the number of times to use the base as a factor.

Definition

If a is a real number and n is a positive integer, the expression a^n represents the product of n factors of a. That is,

$$a^n = \underbrace{a \cdot a \cdot \cdots \cdot a}_{n \text{ factors}}$$

Classroom Example 1
Evaluate each expression.

(a) 3^2 (b) $\left(\frac{5}{2}\right)^3$ (c) 4^4

Answer:
(a) 9 (b) $\frac{125}{8}$ (c) 256

Thus, we have

$$a^1 = a$$
$$a^2 = a \cdot a$$
$$a^3 = a \cdot a \cdot a$$

and so on.

a^n is read as "a raised to the power n" or as "a to the nth power." a^2 is usually read as "a squared" and a^3 as "a cubed."

EXAMPLE 1 Evaluating Expressions Containing Exponents

Evaluate each expression.

(a) 2^3 **(b)** $\left(\frac{1}{2}\right)^2$ **(c)** 5^4

Solution

(a) The expression 2^3 means that the base, 2, is used as a factor 3 times, so
$2^3 = 2 \cdot 2 \cdot 2 = 8$

(b) $\left(\frac{1}{2}\right)^2 = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$ **(c)** $5^4 = 5 \cdot 5 \cdot 5 \cdot 5 = 625$

EXAMPLE 2 Evaluating Expressions Containing Exponents

Classroom Example 2
Evaluate each expression.
(a) $(-2)^2$ (b) $(-2)^3$
(c) -2^2 (d) $-(-2)^5$

Answer:
(a) 4 (b) −8 (c) −4 (d) 32

Evaluate each expression.

(a) $(-3)^4$ **(b)** $(-3)^5$ **(c)** -3^4 **(d)** $-(-3)^4$

Solution

(a) $(-3)^4 = (-3)(-3)(-3)(-3) = 81$

(b) $(-3)^5 = (-3)(-3)(-3)(-3)(-3) = -243$

(c) The expression -3^4 means the additive inverse, or opposite, of 3^4. That is,
$-3^4 = -(3)(3)(3)(3) = -81.$

(d) $-(-3)^4 = -(-3)(-3)(-3)(-3) = -81$

Teaching Tip
Emphasize the importance of parentheses, i.e., the difference between -2^4 and $(-2)^4$.

Work Smart

The expressions $(-3)^4$ and -3^4 are different. $(-3)^4$ means to use (-3) as a factor four times, while -3^4 means to determine the additive inverse of 3^4. That is,

$$(-3)^4 = (-3)(-3)(-3)(-3) = 81$$

and

$$-3^4 = -(3)(3)(3)(3) = -81$$

Another way to look at -3^4 is $-1 \cdot 3^4$.

The results of Examples 2(a) and 2(b) suggest that raising a negative number to an even integer exponent yields a positive number, while raising a negative number to an odd integer exponent yields a negative number. This conclusion is correct in general.

Quick ✓

1. In the expression 5^6, the number 5 is the base and 6 is the exponent or power.
2. *True or False* $-7^4 = (-7)(-7)(-7)(-7)$ False

In Problems 3–8, evaluate each expression.

3. 4^3 64 4. $(-7)^2$ 49 5. $(-10)^3$ -1000 6. $\left(\frac{2}{3}\right)^3$ $\frac{8}{27}$ 7. -8^2 -64 8. $-(-5)^3$ 125

2 Use the Order of Operations to Evaluate Expressions

To evaluate an expression containing both multiplication and addition, such as $2 + 3 \cdot 7$, order of operation dictates the following:

If the two operations of addition and multiplication separate three numbers, always multiply first, then add.

Therefore, $2 + 3 \cdot 7 = 2 + 21 = 23$.

EXAMPLE 3 Finding the Value of an Expression

Classroom Example 3
Evaluate each expression.
(a) $6 + 2 \cdot 7$ (b) $3 \cdot 5 + 8$
Answer: (a) 20 (b) 23

Evaluate each expression.

(a) $3 + 4 \cdot 5$ **(b)** $8 \cdot 2 + 1$

Solution

(a) Don't forget to use the order of operations. Multiply before adding.

$$\begin{aligned} 3 + 4 \cdot 5 &= 3 + 20 \\ &= 23 \end{aligned}$$

(b) $$\begin{aligned} 8 \cdot 2 + 1 &= 16 + 1 \\ &= 17 \end{aligned}$$

Quick ✓

In Problems 9 and 10, evaluate each expression.

9. $5 \cdot 2 + 6$ 16 10. $3 \cdot 2 + 5 \cdot 6$ 36

When addition is to take place before multiplication, parentheses will be used. For example, to evaluate $(2 + 3) \cdot 7$, add 2 and 3 first, then multiply the sum by 7.

EXAMPLE 4 Finding the Value of an Expression

Classroom Example 4
Evaluate each expression.
(a) $-2(6 + 7)$
(b) $(4 + 5)(7 - 3)$
Answer: (a) -26 (b) 36

(a) $$\begin{aligned} (6 + 2) \cdot 4 &= 8 \cdot 4 \\ &= 32 \end{aligned}$$

(b) $$\begin{aligned} (8 - 3)(7 + 3) &= 5 \cdot 10 \\ &= 50 \end{aligned}$$

Quick ✓

In Problems 11–14, evaluate each expression.

11. $4(5 + 3)$ 32 12. $8(9 - 3)$ 48
13. $(12 - 4)(18 - 13)$ 40 14. $(4 + 9)(6 - 4)$ 26

Work Smart

Do not use the Reduction Property across addition.
This is incorrect:

$$\frac{4+5}{6+12} = \frac{\cancel{2}\cdot 2+5}{\cancel{2}\cdot 3+12} = \frac{2+5}{3+12} = \frac{7}{15}$$

This is correct:

$$\frac{4+5}{6+12} = \frac{9}{18} = \frac{1}{2}$$

In expressions like $\frac{4+5}{6+12}$, it is understood that the division bar acts like parentheses. This means evaluate the expressions in the numerator and denominator FIRST, and then divide. That is,

$$\frac{4+5}{6+12} = \frac{(4+5)}{(6+12)} = \frac{9}{18} = \frac{\cancel{9}\cdot 1}{\cancel{9}\cdot 2} = \frac{1}{2}$$

This rule will be extremely important throughout the course.

In addition to parentheses, grouping symbols include brackets [ ], braces { }, and absolute value symbols | |.

If one set of grouping symbols is embedded within a second set of grouping symbols, as in $2 + [3 + 2(9 - 4)]$, simplify within the innermost grouping symbols first.

EXAMPLE 5

Finding the Value of an Expression

Classroom Example 5
Evaluate each expression.
(a) $\frac{8+7}{6\cdot 5-3}$
(b) $2 + [6 - 4(3 - 5)]$
Answer: (a) $\frac{5}{9}$ (b) 16

Evaluate each expression.

(a) $\frac{3+7}{6+4\cdot 6}$ **(b)** $9[8 + 4(10 - 7)]$

Solution

(a)
$$\frac{3+7}{6+4\cdot 6} = \frac{3+7}{6+24} = \frac{10}{30} = \frac{1}{3}$$

(b) Evaluate the expression in the innermost grouping symbols first.

$$\begin{aligned} 9[8+4(10-7)] &= 9[8+4(3)] \\ &= 9[8+12] \\ &= 9\cdot 20 \\ &= 180 \end{aligned}$$

Work Smart

When one set of parentheses is embedded within another set of parentheses, simplify the expression in the innermost parentheses first.

Quick ✓

In Problems 15–18, evaluate each expression.

15. $\frac{3+7}{4+9}$ $\frac{10}{13}$

16. $1 - 4 + 8\cdot 2 + 5$ 18

17. $25[2(8 - 3) - 8]$ 50

18. $\frac{3+5}{2(9-4)}$ $\frac{4}{5}$

Order of Operations with Exponents

When are exponents evaluated in the order of operations? In the expression $3\cdot 2^4$, do we multiply 3 and 2 first and then raise this product to the 4th power to get 1296, or do we first raise 2 to the 4th power to get 16 and then multiply this by 3 to obtain 48? Because $2^4 = 2\cdot 2\cdot 2\cdot 2$, we have $3\cdot 2^4 = 3\cdot 2\cdot 2\cdot 2\cdot 2 = 48$, which indicates that exponents should be evaluated before multiplication.

Teaching Tip
Students may be familiar with the expression "Please Excuse My Dear Aunt Sally" as a mnemonic device to remember the order of operations. Remind students that "P" (parentheses) is not an operation, just a guide to remembering where to begin evaluating an expression. In addition, this mnemonic device may mislead students into thinking they should always multiply before dividing, or always add before subtracting.

Order of Operations

1. Evaluate expressions within **parentheses** first. When an expression has more than one set of grouping symbols, begin with the innermost grouping symbols and work outward.
2. Evaluate expressions containing **exponents,** working from left to right.
3. Perform **multiplication** and **division,** working from left to right.
4. Perform **addition** and **subtraction,** working from left to right.

Quick ✓

19. The order of operations is (1) parentheses, (2) exponents, (3) multiplication and division, (4) addition and subtraction.

In Problems 20–24, evaluate each expression.

20. $6 + 5 \cdot 2$ 16 **21.** $(3 + 9) \cdot 4$ 48 **22.** $\dfrac{7 + 5}{4 + 10}$ $\dfrac{6}{7}$

23. $4 + [(8 - 3) \cdot 2]$ 14 **24.** $4(6 - 2) - 9$ 7

EXAMPLE 6 Evaluating Expressions Using the Order of Operations

Classroom Example 6
Evaluate each expression.
(a) $3(-2)^2 + 4(7 - 2^3)$
(b) $\dfrac{-1 \cdot 12 + 2(-1)^2}{3(5 - 9)}$
Answer: (a) 8 (b) $\dfrac{5}{6}$

Evaluate each expression.

(a) $6 + 4 \cdot 3^2$ **(b)** $\dfrac{1}{4} \cdot 2^3 + 9 \cdot 4 - 2 \cdot 5^2$

Solution

(a) Evaluate the exponent first:

$$\begin{aligned} 6 + 4 \cdot 3^2 &= 6 + 4 \cdot 9 \\ \text{Multiply:} \quad &= 6 + 36 \\ &= 42 \end{aligned}$$

(b)
$$\begin{aligned} \frac{1}{4} \cdot 2^3 + 9 \cdot 4 - 2 \cdot 5^2 &= \frac{1}{4} \cdot 8 + 9 \cdot 4 - 2 \cdot 25 \\ \text{Multiply:} \quad &= 2 + 36 - 50 \\ \text{Add from left to right:} \quad &= 38 - 50 \\ &= -12 \end{aligned}$$

EXAMPLE 7 Evaluating Expressions Using the Order of Operations

Classroom Example 7
Evaluate each expression.
(a) $\dfrac{8^2 - 6 \cdot 5}{2 - |12 - 4|}$
(b) $3 - 4[7^2 + (3)(-5)]$
Answer: (a) $-\dfrac{17}{3}$ (b) -133

Evaluate each expression.

(a) $\dfrac{2 + 3 \cdot 4^2}{5(6 - 2)}$ **(b)** $-2|-(-2)^3 - 4(10 - 7)^2|$

Solution

(a)
$$\begin{aligned} \frac{2 + 3 \cdot 4^2}{5(6 - 2)} &= \frac{2 + 3 \cdot 16}{5 \cdot 4} \\ &= \frac{2 + 48}{20} \\ &= \frac{50}{20} \end{aligned}$$

$$= \frac{5\cdot 10}{2\cdot 10}$$

Divide out the common factor, 10: $= \dfrac{5}{2}$

(b) The absolute value bars form a grouping symbol, so simplify within the absolute value bars first.

Evaluate inner parentheses first:

$$\begin{aligned}
-2|-(-2)^3 - 4(10-7)^2| &= -2|-(-2)^3 - 4(3)^2| \\
\text{Evaluate exponents:}\quad &= -2|-(-8) - 4\cdot 9| \\
&= -2|8 - 36| \\
&= -2|-28| \\
&= -2\cdot 28 \\
&= -56
\end{aligned}$$

Note to Instructor
Quick Check exercises are designed as homework. For best results, assign them.

Quick ✓

In Problems 25–30, evaluate each expression.

25. $-8 + 2\cdot 5^2$ 42
26. $5\cdot 3 - 3\cdot 2^3$ -9
27. $5(10-8)^2$ 20
28. $3(-2)^2 + 6\cdot 3 - 3\cdot 4^2$ -18
29. $\dfrac{4+6^2}{2\cdot 3 + 2}$ 5
30. $-3|7^2 - 2(8-5)^3|$ -15

R.4 Exercises

MyLabMath® Underlined exercise numbers have complete video solutions in MyLabMath or may be accessed using the QR code to the right.

Problems **1–30** *are the* **Quick ✓s** *that follow the* **EXAMPLES.**

Building Skills

In Problems 31–38, evaluate each expression. See Objective 1.

31. $(-3)^4$ 81
32. $(-4)^3$ -64
33. -5^4 -625
34. -2^4 -16
35. $-(-2)^3$ 8
36. $-(-7)^3$ 343
37. $-\left(\dfrac{2}{3}\right)^2$ $-\dfrac{4}{9}$
38. $-\left(\dfrac{3}{4}\right)^4$ $-\dfrac{81}{256}$

Mixed Practice

In Problems 39–74, evaluate each expression.

39. $4^2 - 3^2$ 7
40. $(-3)^2 + (-2)^2$ 13
41. $3\cdot 2 + 9$ 15
42. $-5\cdot 3 + 12$ -3
43. $4 + 2(6-2)$ 12
44. $2 + 5(8-5)$ 17
45. $-2[10 - (3-7)]$ -28
46. $3[15 - (7-3)]$ 33
47. $\dfrac{5-(-7)}{4}$ 3
48. $\dfrac{12-4}{-2}$ -4
49. $|3\cdot 2 - 4\cdot 5|$ 14
50. $|6\cdot 2 - 5\cdot 3|$ 3
51. $12\cdot\dfrac{2}{3} - 5\cdot 2$ -2
52. $15\cdot\dfrac{3}{5} + 4\cdot 3$ 21
53. $3[2 + 3(1+5)]$ 60
54. $2[25 - 2(10-4)]$ 26
55. $(3^2 - 3)(3 - (-3)^3)$ 180
56. $-2(5-2) - (-5)^2$ -31
57. $|3(6 - 3^2)|$ 9
58. $-2(4 + |2\cdot 3 - 5^2|)$ -46
59. $|4[2\cdot 5 + (-3)\cdot 4]|$ 8
60. $|6(3\cdot 2 - 10)|$ 24
61. $\dfrac{2\cdot 5 + 15}{2^2 + 3\cdot 2}$ $\dfrac{5}{2}$
62. $\dfrac{2\cdot 3^2}{4^2 - 4}$ $\dfrac{3}{2}$
63. $\dfrac{2(4+8)}{3+3^2}$ 2
64. $\dfrac{3(5+2^2)}{2\cdot 3^3}$ $\dfrac{1}{2}$
65. $\dfrac{6[12 - 3(5-2)]}{5[21 - 2(4+5)]}$ $\dfrac{6}{5}$
66. $\dfrac{4[3 + 2(8-6)]}{5[14 - 2(2+3)]}$ $\dfrac{7}{5}$
67. $\left(\dfrac{2}{3}\right)^2\left(\dfrac{1+2^3}{2^3-2}\right)$ $\dfrac{2}{3}$
68. $\left(\dfrac{3^2}{29 - 3\cdot 2^3}\right)\cdot\dfrac{5}{4+5}$ 1
69. $\dfrac{3^3 - 2^4\cdot 3}{4(3^2 - 2\cdot 3)}$ $-\dfrac{7}{4}$
70. $2^5 + 4(-5) + 4^2$ 28
71. $\dfrac{2\cdot 4 - 5}{4^2 + (-2)^3} + \dfrac{3^2}{2^3}$ $\dfrac{3}{2}$
72. $\dfrac{5(37 - 6^2)}{6\cdot 2 - 3^2} + \dfrac{7\cdot 2 - 4^2}{5+4}$ $\dfrac{13}{9}$
73. $\dfrac{\dfrac{2\cdot 3 + 3^2}{2\cdot 5 - 8} + \dfrac{4}{3}}{\dfrac{7(5-3)}{14 - 2^3}}$ $\dfrac{53}{14}$
74. $\dfrac{\dfrac{4}{4^2 - 1} - \dfrac{3}{5(7-5)}}{\dfrac{-(-2)^2}{4\cdot 7 + 2}}$ $\dfrac{1}{4}$

Applying the Concepts

In Problems 75–78, insert parentheses in order to make each statement true.

75. $3 \cdot 7 - 2 = 15$ $\quad 3 \cdot (7 - 2) = 15$

76. $-2 \cdot 3 - 5 = 4$ $\quad -2 \cdot (3 - 5) = 4$

77. $3 + 5 \cdot 6 - 3 = 18$ $\quad 3 + 5 \cdot (6 - 3) = 18$

78. $3 + 5 \cdot 6 - 3 = 24$ $\quad (3 + 5) \cdot (6 - 3) = 24$

△ **79. Geometry** The surface area of a closed right circular cylinder whose radius is 5 inches and height is 12 inches is given approximately by $2 \cdot 3.1416 \cdot 5^2 + 2 \cdot 3.1416 \cdot 5 \cdot 12$. Evaluate this expression, rounding your answer to two decimal places. 534.07 in.^{2}

△ **80. Geometry** The surface area of a sphere whose radius is 3 centimeters is given approximately by $4 \cdot 3.1416 \cdot 3^2$. Evaluate this expression, rounding your answer to two decimal places. 113.10 cm.^{2}

81. Hitting a Golf Ball The height (in feet) of a golf ball hit with an initial speed of 100 feet per second after 3 seconds is given by $-16 \cdot 3^2 + 50 \cdot 3$. Evaluate this expression in order to determine the height of the golf ball after 3 seconds. 6 feet

82. Horsepower The horsepower rating of an engine is $\frac{10^2 \cdot 8}{2.5}$. Evaluate this expression in order to determine the horsepower rating of the engine. 320 hp.

Math for the Future *Problems 83 and 84 show some computations required in statistics. Completely simplify each expression.*

83. $\dfrac{105 + 80 + 115 + 95 + 105}{5}$ $\quad 100$

84. $\dfrac{(105 - 100)^2 + (80 - 100)^2 + (115 - 100)^2 + (95 - 100)^2 + (105 - 100)^2}{4}$ $\quad 175$

Explaining the Concepts

85–88. See Graphing Answer Section.

85. Explain why $\dfrac{2 + 7}{2 + 9} \neq \dfrac{7}{9}$.

86. Develop an example that illustrates why we perform multiplication before addition when there are no grouping symbols.

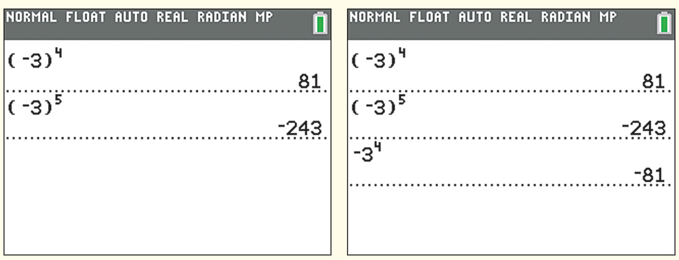
87. Develop an example that illustrates why we evaluate exponents before multiplication when there are no grouping symbols.

88. Explain the difference between -4^3 and $(-4)^3$.

Technology Exercises

Calculators can be used to evaluate exponential expressions. Figures 11(a) and (b) show the results of Examples 2(a), (b), and (c) using a graphing calculator.

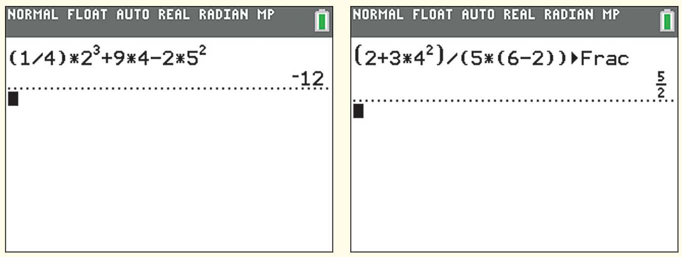
Figure 11



(a) (b)

Graphing calculators know the order of operations. Figure 12(a) shows the result of Example 6(b), and Figure 12(b) the result of Example 7(a), using a graphing calculator. Be careful with the placement of parentheses when using the calculator!

Figure 12



(a) (b)

In Problems 89–96, use technology to evaluate each expression. When necessary, express answers rounded to two decimal places.

89. $\dfrac{4}{5} - \left(\dfrac{2}{3}\right)^2$ $\quad \dfrac{16}{45}$

90. $3 - \left(\dfrac{6}{5}\right)^3$ $\quad \dfrac{159}{125}$

91. $\dfrac{4^2 + 1}{13}$ $\quad \dfrac{17}{13}$

92. $\dfrac{3^2 - 2^3}{1 + 3 \cdot 2}$ $\quad \dfrac{1}{7}$

93. $\dfrac{3.5^3}{1.3^2} - 6.2^3$ $\quad -212.96$

94. $2.3^4 \cdot \dfrac{4}{11} - (3.7)^2 \cdot \dfrac{8}{3}$ $\quad -26.33$

95. $4.3[9.3^2 - 4(34.2 + 18.5)]$ $\quad -534.53$

96. $6.3^2 + 4.2^2$ $\quad 57.33$

R.5 Algebraic Expressions

Objectives

1. Translate English Expressions into Mathematical Language
2. Evaluate Algebraic Expressions
3. Simplify Algebraic Expressions by Combining Like Terms
4. Determine the Domain of a Variable

In Other Words
In algebra, letters of the alphabet are used to represent numbers.

Remember that if a letter is used to represent *any* number from a given set of numbers, it is called a **variable.** A **constant** is either a fixed number, such as 5 or $\sqrt{2}$, or a letter that represents a fixed (possibly unspecified) number. For example, in Einstein's Theory of Relativity, $E = mc^2$, E and m are variables that represent total energy and mass, respectively, while c is a constant that represents the speed of light (299,792,458 meters per second).

An **algebraic expression** is any combination of variables, constants, grouping symbols such as parentheses ( ) and brackets [ ], and mathematical operations such as addition, subtraction, multiplication, division, and exponents. The following are examples of algebraic expressions.

$$3x + 4 \qquad 2y^2 - 5y - 12z \qquad \frac{5v^4 - v}{4 - v}$$

1 Translate English Expressions into Mathematical Language

One feature of mathematics is that various English phrases can be translated into a few math symbols. Table 2 lists some English words and phrases and their corresponding math symbols.

Work Smart
An algebraic expression is not the same as an algebraic equation. An expression might be $x + 3$ while an equation might be $x + 3 = 7$. Do you see the difference?

Table 2 Math Symbols and the Words They Represent

Add (+)	Subtract (−)	Multiply (·)	Divide (/)
sum	difference	product	quotient
plus	minus	times	divided by
more than	subtracted from	of	per
exceeds by	less	twice	ratio
added to	decreased by		
increased by			

EXAMPLE 1 Writing English Phrases Using Math Symbols

Express each English phrase as an algebraic expression.

(a) The sum of 3 and 8
(b) The quotient of 50 and some number y
(c) 9 less 15
(d) Twice the sum of a number x and 5

Classroom Example 1
Express each English phrase as an algebraic expression.
(a) The sum of a number n and 5
(b) The difference of a number k and 7
(c) The sum of twice a number x and 5
(d) The quotient of a number y and 3

Answer: (a) $n + 5$ (b) $k - 7$ (c) $2x + 5$ (d) $\frac{y}{3}$

Solution

(a) A sum implies the $+$ symbol, so "the sum of 3 and 8" is represented mathematically as $3 + 8$.

(b) "The quotient of 50 and some number y" is represented mathematically as $\frac{50}{y}$.

(c) "9 less 15" is represented mathematically as $9 - 15$.

(d) "Twice the sum of a number x and 5" is represented as $2(x + 5)$. Notice that the example said "twice the sum." (The English phrase represented as $2x + 5$ might be "the sum of twice a number and 5." Do you see the difference?)

Teaching Tip
Point out that the word "evaluate" contains the word VALUE—to evaluate an expression means to find its numerical value.

Quick ✓

1. A variable is a letter used to represent any number from a given set of numbers.
2. A constant is either a fixed number or a letter used to represent a fixed (possibly unspecified) number.

In Problems 3–8, express each English phrase using an algebraic expression.

3. The sum of 3 and 11 $3 + 11$
4. The product of 6 and 7 $6 \cdot 7$
5. The quotient of y and 4 $\frac{y}{4}$
6. The difference of 3 and z $3 - z$
7. Twice the difference of x and 3 $2(x - 3)$
8. The difference of twice a number x and 3 $2x - 3$

2 Evaluate Algebraic Expressions

To **evaluate an algebraic expression,** substitute a numerical value for each variable in the expression, and simplify the result.

EXAMPLE 2 Evaluating an Algebraic Expression

Classroom Example 2
Evaluate each expression for the given value of the variable.
(a) $5z - 3$ for $z = 2$
(b) $\frac{4n - n^2}{n + 2}$ for $n = -5$
(c) $|3y - 5|$ for $y = \frac{2}{3}$
Answer: (a) 7 (b) 15 (c) 3

Work Smart
Be sure to use parentheses when you substitute a number in for a variable raised to an exponent.

Evaluate each expression for the given value of the variable.

(a) $4x + 3$ for $x = 5$ **(b)** $\frac{-z^2 + 4z}{z + 1}$ for $z = 9$ **(c)** $|10y - 8|$ for $y = \frac{1}{2}$

Solution

(a) Substitute 5 for x in the expression $4x + 3$.

$$4(5) + 3 = 20 + 3 = 23$$

(b) Substitute 9 for z in the expression $\frac{-z^2 + 4z}{z + 1}$.

$$\frac{-(9)^2 + 4 \cdot 9}{9 + 1} = \frac{-81 + 36}{10} = \frac{-45}{10} = \frac{-9 \cdot \cancel{5}}{\cancel{5} \cdot 2} = \frac{-9}{2} = -\frac{9}{2}$$

(c) Substitute $\frac{1}{2}$ for y in the expression $|10y - 8|$.

$$\left|10 \cdot \frac{1}{2} - 8\right| = |5 - 8| = |-3| = 3$$

Classroom Example 3
The algebraic expression $2x + 2(x + 6)$ represents the perimeter of a rectangle whose width is 6 inches more than the length, x. Evaluate the algebraic expression for $x = 5$, 8, and 12.
Answer: 32 inches, 44 inches, 60 inches

Quick ✓

9. To evaluate an algebraic expression, substitute a numerical value for each variable in the expression, and simplify the result.

In Problems 10–13, evaluate each expression for the given value of the variable.

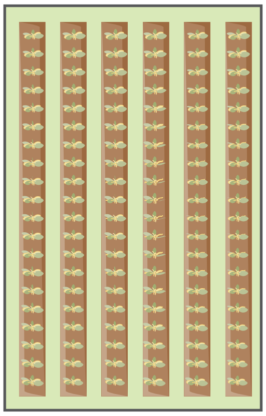
10. $-5x + 3$ for $x = 2$ -7
11. $y^2 - 6y + 1$ for $y = -4$ 41
12. $\frac{w + 8}{3w}$ for $w = 4$ 1
13. $|4x - 5|$ for $x = \frac{1}{2}$ 3

EXAMPLE 3 Evaluating an Algebraic Expression

The algebraic expression $2x + 2(x + 10)$ represents the perimeter of a rectangular field whose length is 10 yards more than its width. See Figure 13 on the next page. Evaluate the algebraic expression for $x = 4$, 8, and 10.

Figure 13

$x + 10$



x

Solution

Evaluate the algebraic expression $2x + 2(x + 10)$ for each value of x.

$$x = 4: \quad 2(4) + 2(4 + 10) = 8 + 2(14) = 8 + 28 = 36 \text{ yards}$$
$$x = 8: \quad 2(8) + 2(8 + 10) = 16 + 2(18) = 16 + 36 = 52 \text{ yards}$$
$$x = 10: \quad 2(10) + 2(10 + 10) = 20 + 2(20) = 20 + 40 = 60 \text{ yards}$$

Quick ✓

14. The algebraic expression $113x$ approximates the number of Japanese yen that you could purchase for x dollars. Evaluate the algebraic expression for $x = 100$, 1000, and 10,000 dollars. (Source: *Yahoo! Finance*)
11,300 yen; 113,000 yen; 1,130,000 yen

15. The algebraic expression $\frac{5}{9}(x - 32)$ represents the temperature in degrees Celsius that is equivalent to x degrees Fahrenheit. Determine the equivalent temperature for $x = 32$, 86, and 212 degrees Fahrenheit. 0°C; 30°C; 100°C

3 Simplify Algebraic Expressions by Combining Like Terms

One way to simplify algebraic expressions is to combine *like terms*. A **term** is a number or the product of a number and one or more variables raised to a power. In algebraic expressions, terms are separated by addition signs. See Table 3, which lists the terms in various algebraic expressions.

Table 3

Algebraic Expression	Terms
$5x + 4$	$5x$, 4
$7x^2 - 8x + 3 = 7x^2 + (-8x) + 3$	$7x^2, -8x, 3$
$3x^2 + 7y^2$	$3x^2, 7y^2$

Notice that in the algebraic expression $7x^2 - 8x + 3$, we first rewrite it with only addition signs as $7x^2 + (-8x) + 3$ to identify the terms.

Terms that have the same variable(s) and the same exponent(s) on the variable(s) are **like terms.** For example, $3x$ and $8x$ are like terms because both terms have the variable x raised to the 1st power. Also, $-4x^3y$ and $10x^3y$ are like terms because both have the variable x raised to the 3rd power, along with y raised to the 1st power. The numerical factor of the term is the **coefficient.** For example, the coefficient of $3x$ is 3; the coefficient of $-4x^3y$ is -4. Terms that have no number as a factor, such as xy, have the coefficient one (1) because $xy = 1 \cdot xy$. The coefficient of the term $-z$ is negative one (-1) because $-z = -1 \cdot z$. If a term consists of just a constant, the coefficient is the number itself. For example, the coefficient of the term 19 is 19.

To combine like terms use the Distributive Property "in reverse."

EXAMPLE 4 Combining Like Terms

Classroom Example 4
Simplify each algebraic expression by combining like terms.
(a) $2n + 7n$ (b) $r + 5r - 7$
Answer: (a) $9n$ (b) $6r - 7$

Simplify each algebraic expression by combining like terms.

(a) $5x + 3x$ **(b)** $z + 7z - 5$

Solution

(a) $5x + 3x = (5 + 3)x = 8x$

(b) Since $z = 1 \cdot z$ because of the multiplicative identity, it follows that

$$z + 7z - 5 = 1z + 7z - 5$$

Use the Distributive Property "in reverse": $= (1 + 7)z - 5$

Combine like terms: $= 8z - 5$

Quick ✓

16. Terms that have the same variable(s) and the same exponent(s) on the variables are called like terms.

17. The coefficient of the term $-m$ is -1.

In Problems 18–21, simplify each expression by combining like terms.

18. $4x - 9x$ $\quad -5x$

19. $-2x^2 + 13x^2$ $\quad 11x^2$

20. $-5x - 3x + 6 - 3$ $\quad -8x + 3$

21. $6x - 10x - 4y + 12y$ $\quad -4x + 8y$

Sometimes terms must be rearranged using the Commutative Property of Addition to combine like terms.

EXAMPLE 5 Combining Like Terms Using the Commutative and Distributive Properties

Classroom Example 5
Simplify each algebraic expression.
(a) $8y^2 - 3 + 2y^2 + 11$
(b) $5k + 6 + 4k - 10k - 5$
Answer: (a) $10y^2 + 8$ (b) $-k + 1$

Simplify each algebraic expression.

(a) $13y^2 + 8 - 4y^2 + 3$ **(b)** $12z + 5 + 5z - 8z - 2$

Solution

(a) Use the Commutative Property to rearrange the terms.

$$13y^2 + 8 - 4y^2 + 3 = 13y^2 - 4y^2 + 8 + 3$$

Use the Distributive Property "in reverse": $= (13 - 4)y^2 + 8 + 3$

Combine like terms: $= 9y^2 + 11$

(b) Rearrange terms before using the Distributive Property.

$$12z + 5 + 5z - 8z - 2 = 12z + 5z - 8z + 5 - 2$$

Use the Distributive Property "in reverse": $= (12 + 5 - 8)z + 5 - 2$

Combine like terms: $= 9z + 3$

Quick ✓

In Problems 22–24, simplify each expression by combining like terms.

22. $10y - 3 + 5y + 2$ $\quad 15y - 1$

23. $0.5x^2 + 1.3 + 1.8x^2 - 0.4$ $\quad 2.3x^2 + 0.9$

24. $4z + 6 - 8z - 3 - 2z$ $\quad -6z + 3$

It may also be necessary to remove parentheses using the Distributive Property before combining like terms.

EXAMPLE 6 Combining Like Terms Using the Distributive Property

Classroom Example 6
Simplify each algebraic expression.
(a) $4(x - 2) - x$
(b) $7p + 5 - 3(p - 6)$
Answer: (a) $3x - 8$ (b) $4p + 23$

Simplify each algebraic expression.

(a) $5(x - 3) - 2x$ **(b)** $x + 4 - 2(x + 3)$

Solution

(a) First, use the Distributive Property to remove the parentheses.

$$5(x - 3) - 2x = 5x - 15 - 2x$$

Rearrange terms: $= 5x - 2x - 15$

Combine like terms: $= 3x - 15$

(b) Distribute -2 to remove the parentheses.

$$x + 4 - 2(x + 3) = x + 4 - 2x - 6$$

Rearrange terms: $= x - 2x + 4 - 6$

Combine like terms: $= -x - 2$

EXAMPLE 7

Combining Like Terms Using the Distributive Property

Simplify each algebraic expression.

(a) $5(x - 3) - (7x - 4)$ **(b)** $\frac{1}{2}(4x + 3) + \frac{8x - 3}{5}$

Classroom Example 7
Simplify each algebraic expression.
(a) $2(3m + 4) - (8m + 2)$
(b) $\frac{2}{3}(6x + 4) + \frac{2x - 1}{6}$
Answer: (a) $-2m + 6$
(b) $\frac{13}{3}x + \frac{5}{2}$

Solution

(a) When grouping symbols are preceded by a minus sign, multiply the terms in the grouping symbol by -1.

$$5(x - 3) - (7x - 4) = 5(x - 3) - 1(7x - 4)$$

Use the Distributive Property: $= 5x - 15 - 7x + 4$

Rearrange terms: $= 5x - 7x - 15 + 4$

Combine like terms: $= -2x - 11$

(b) Use the fact that $\frac{a}{b} = \frac{1}{b} \cdot a$, so $\frac{8x - 3}{5} = \frac{1}{5}(8x - 3)$.

$$\frac{1}{2}(4x + 3) + \frac{8x - 3}{5} = \frac{1}{2}(4x + 3) + \frac{1}{5}(8x - 3)$$

Use the Distributive Property: $= \frac{1}{2} \cdot 4x + \frac{1}{2} \cdot 3 + \frac{1}{5} \cdot 8x + \frac{1}{5} \cdot (-3)$

Simplify: $= 2x + \frac{3}{2} + \frac{8}{5}x - \frac{3}{5}$

Rearrange terms; rewrite fractions with least common denominator: $= \frac{10}{5}x + \frac{8}{5}x + \frac{15}{10} - \frac{6}{10}$

Combine like terms: $= \frac{18}{5}x + \frac{9}{10}$

Quick ✓

In Problems 25–30, simplify each expression by combining like terms.

25. $3(x - 2) + x$ $4x - 6$

26. $5(y + 3) - 10y - 4$ $-5y + 11$

27. $3(z + 4) - 2(3z + 1)$ $-3z + 10$

28. $-4(x - 2) - (2x + 4)$ $-6x + 4$

29. $\frac{1}{2}(6x + 4) - \frac{15x + 5}{5}$ 1

30. $\frac{5x - 1}{3} + \frac{5x + 9}{2}$ $\frac{25x + 25}{6} = \frac{25(x + 1)}{6}$

4 Determine the Domain of a Variable

In some algebraic expressions, the variable may be allowed to take on values from only a certain set of numbers. For example, in the expression $\frac{1}{x}$, the variable x cannot take on the value 0, because this would cause division by 0, which is not defined.

Definition

The set of values that a variable may assume is called the **domain of the variable.**

EXAMPLE 8 Determining the Domain of a Variable

Classroom Example 8
Determine which of the following numbers are in the domain of the variable x for the expression $\frac{4}{x-5}$.
(a) $x = 7$ (b) $x = -3$
(c) $x = 5$ (d) $x = 0$
Answer: (a) yes (b) yes
(c) no (d) yes

Determine which of the following numbers are in the domain of the variable x for the expression $\frac{4}{x+3}$.

(a) $x = 3$ **(b)** $x = 0$ **(c)** $x = -3$

Solution

Determine whether the value of the variable causes division by 0. That is, determine whether the value of the variable causes $x + 3$ to equal 0. If it does, exclude it from the domain.

(a) When $x = 3$, the denominator is $x + 3 = 3 + 3 = 6$, so 3 is in the domain of the variable.

(b) When $x = 0$, the denominator is $x + 3 = 0 + 3 = 3$, so 0 is in the domain.

(c) When $x = -3$, the denominator $x + 3 = -3 + 3 = 0$, so -3 is NOT in the domain.

Note to Instructor
Quick Check exercises are designed as homework. For best results, assign them.

Quick ✓

31. The set of values that a variable may assume is called the domain of the variable.

In Problems 32–34, determine which of the following numbers are in the domain of the variable.

(a) $x = 2$ **(b)** $x = 0$ **(c)** $x = 4$ **(d)** $x = -3$

32. $\frac{2}{x-4}$ All except $x = 4$

33. $\frac{x}{x+3}$ All except $x = -3$

34. $\frac{x+3}{x^2+x-6}$ $x = 0; x = 4$

R.5 Exercises

MyLabMath® Underlined exercise numbers have complete video solutions in MyLabMath or may be accessed using the QR code to the right.

*Problems **1–34** are the **Quick ✓s** that follow the **EXAMPLES**.*

Building Skills

In Problems 35–46, express each English phrase using an algebraic expression. See Objective 1.

35. The sum of 5 and a number x $5 + x$

36. The difference of 10 and a number y $10 - y$

37. The product of 4 and a number z $4z$

38. The ratio of a number x and 5 $\frac{x}{5}$

39. A number y decreased by 7 $y - 7$

40. A number z increased by 30 $z + 30$

41. Twice the sum of a number t and 4 $2(t + 4)$

42. The sum of a number x and 5 divided by 10 $\frac{x+5}{10}$

43. Three less than five times a number x $5x - 3$

44. Three times a number z increased by the quotient of z and 8 $3z + \frac{z}{8}$

45. The quotient of some number y and 3 increased by the product of 6 and some number x $\frac{y}{3} + 6x$

46. Twice some number x decreased by the ratio of a number y and 3 $2x - \frac{y}{3}$

In Problems 47–62, evaluate each expression for the given value of the variable. See Objective 2.

47. $4x + 3$ for $x = 2$ 11

48. $-5x + 1$ for $x = -3$ 16

49. $x^2 + 5x - 3$ for $x = -2$ -9

50. $y^2 - 4y + 5$ for $y = 3$ 2

51. $4 - z^2$ for $z = -5$ -21

52. $3 + z - 2z^2$ for $z = -4$ -33

53. $\frac{2w}{w^2 + 2w + 1}$ for $w = 3$ $\frac{3}{8}$

54. $\frac{4z + 3}{z^2 - 4}$ for $z = 3$ 3

55. $\dfrac{v^2+2v+1}{v^2+3v+2}$ for $v=5$ $\quad\dfrac{6}{7}$

56. $\dfrac{2x^2+5x+2}{x^2+5x+6}$ for $x=3$ $\quad\dfrac{7}{6}$

57. $|5x-4|$ for $x=-5$ $\quad 29$

58. $|x^2-6x+1|$ for $x=2$ $\quad 7$

59. $(x+2y)^2$ for $x=3, y=-4$ $\quad 25$

60. $(a-5b)^2$ for $a=1, b=3$ $\quad 196$

61. $\dfrac{(x+2)^2}{|4x-10|}$ for $x=1$ $\quad\dfrac{3}{2}$

62. $\dfrac{|3-5z|}{(z-4)^2}$ for $z=4$ $\quad$ undefined

In Problems 63–92, simplify each expression by combining like terms. See Objective 3.

63. $3x-2x$ $\quad x$

64. $5y+2y$ $\quad 7y$

65. $-4z-2z+3$ $\quad -6z+3$

66. $8x-9x+1$ $\quad -x+1$

67. $13z+2-14z-7$ $\quad -z-5$

68. $-10x+6+4x-x+1$ $\quad -7x+7$

69. $\dfrac{3}{4}x+\dfrac{1}{6}x$ $\quad\dfrac{11}{12}x$

70. $\dfrac{3}{10}y+\dfrac{4}{15}y$ $\quad\dfrac{17}{30}y$

71. $2x+3x^2-5x+x^2$ $\quad 4x^2-3x$

72. $-x-3x^2+4x-x^2$ $\quad -4x^2+3x$

73. $-1.3x-3.4+2.9x+3.4$ $\quad 1.6x$

74. $2.5y-1.8-1.4y+0.4$ $\quad 1.1y-1.4$

75. $3x-2-x+3-5x$ $\quad -3x+1$

76. $10y+3-2y+6+y$ $\quad 9y+9$

77. $-2(5x-4)-(4x+1)$ $\quad -14x+7$

78. $3(2y+5)-6(y+2)$ $\quad 3$

79. $5(z+2)-6z$ $\quad -z+10$

80. $\dfrac{1}{2}(20x-14)+\dfrac{1}{3}(6x+9)$ $\quad 12x-4$

81. $\dfrac{2}{5}(5x-10)+\dfrac{1}{4}(8x+4)$ $\quad 4x-3$

82. $4(w+2)+3(4w+3)$ $\quad 16w+17$

83. $2(v-3)+5(2v-1)$ $\quad 12v-11$

84. $-4(w-3)-(2w+1)$ $\quad -6w+11$

85. $\dfrac{3}{5}(10x+4)-\dfrac{8x+3}{2}$ $\quad 2x+\dfrac{9}{10}$

86. $\dfrac{4}{3}(5y+1)-\dfrac{2}{5}(3y-4)$ $\quad\dfrac{82}{15}y+\dfrac{44}{15}$

87. $\dfrac{5}{6}\left(\dfrac{3}{10}x-\dfrac{2}{5}\right)+\dfrac{2}{3}\left(\dfrac{1}{6}x+\dfrac{1}{2}\right)$ $\quad\dfrac{13}{36}x$

88. $\dfrac{1}{4}\left(\dfrac{2}{3}x-\dfrac{1}{2}\right)+\dfrac{1}{10}\left(\dfrac{5}{2}x-\dfrac{15}{4}\right)$ $\quad\dfrac{5}{12}x-\dfrac{1}{2}$

89. $4.3(1.2x-2.3)+9.3x-5.6$ $\quad 14.46x-15.49$

90. $0.4(2.9x-1.6)-2.7(0.3x+6.2)$ $\quad 0.35x-17.38$

91. $6.2(x-1.4)-5.4(3.2x-0.6)$ $\quad -11.08x-5.44$

92. $9.3(0.2x-0.8)+3.8(1.3x+6.3)$ $\quad 6.8x+16.5$

In Problems 93–98, determine which of the following numbers are in the domain of the variable. See Objective 4.

(a) $x=5$ **(b)** $x=-1$ **(c)** $x=-4$ **(d)** $x=0$

93. $\dfrac{3}{x-5}$ $\quad$ All except $x=5$

94. $\dfrac{7}{x+1}$ $\quad$ All except $x=-1$

95. $\dfrac{x+1}{x+5}$ $\quad$ All

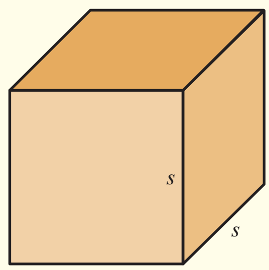
96. $\dfrac{x+4}{x-1}$ $\quad$ All

97. $\dfrac{x}{x^2-5x}$ $\quad x=-1, x=-4$

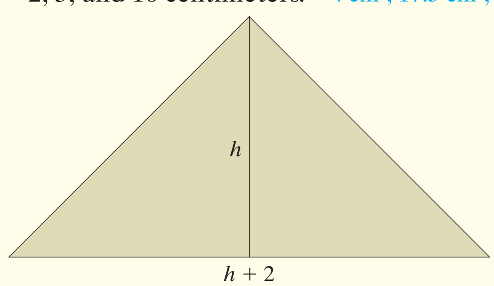
98. $\dfrac{x+1}{x^2+5x+4}$ $\quad x=5, x=0$

Applying the Concepts

△ **99. Volume of a Cube** The algebraic expression s^3 represents the volume of a cube whose sides are length s. See the figure. Evaluate the algebraic expression for $s=1, 2, 3$, and 4 inches.
1 in.^{3}; 8 in.^{3}; 27 in.^{3}; 64 in.^{3}



△ **100. Area of a Triangle** The algebraic expression $\dfrac{1}{2}h(h+2)$ represents the area of a triangle whose base is 2 centimeters longer than its height h. See the figure. Evaluate the algebraic expression for $h=2, 5$, and 10 centimeters. $\quad$ 4 cm^2; 17.5 cm^2; 60 cm^2



101. Projectile Motion The algebraic expression $-16t^2 + 75t$ represents the height (in feet) of a golf ball hit at an angle of 30° to the horizontal after t seconds.

(a) Evaluate the algebraic expression for $t = 0, 1, 2, 3,$ and 4 seconds. 0 ft.; 59 ft.; 86 ft.; 81 ft.; 44 ft.

(b) Use the results of part (a) to describe what happens to the golf ball as time passes. Height increases, reaches a max, then decreases

102. Cost of Production The algebraic expression $30x + 1000$ represents the cost of manufacturing x watches in a day. Evaluate the algebraic expression for $x = 20$, 30, and 40 watches. \$1600; \$1900; \$2200

103. How Old Is Tony? Suppose that Bob is x years of age. Write a mathematical expression for the following: "Tony is 5 years older than Bob." Evaluate the expression if Bob is $x = 13$ years of age. $x + 5$; 18 years

104. Getting a Discount Suppose the regular price of a computer is p dollars. Write a mathematical expression for the following: "I'll give you \$50 off regular price." Evaluate the expression if the regular price of the computer is \$890. $p - 50$; \$840

105. Getting a Big Discount Suppose the regular price of a computer is p dollars. Write a mathematical expression for the following: "I'll give you half off regular price." Evaluate the expression if the regular price of the computer is \$900. $\frac{1}{2}p$; \$450

106. How Old Is Marissa's Mother? Suppose that Marissa is x years of age. Write a mathematical expression for the following: The age of "Marissa's mother is twice the sum of Marissa's age and 3." How old is Marissa's mother if Marissa is $x = 18$ years of age? $2(x + 3)$; 42 years

Math for the Future *For Problems 107 and 108, evaluate the expression* $\frac{X - \mu}{\sigma}$ *for the given values.*

107. $X = 120, \mu = 100, \sigma = 15$ $\quad \frac{4}{3}$

108. $X = 40, \mu = 50, \sigma = 10$ $\quad -1$

109–118. See Graphing Answer Section.

In Problems 109–114, write an English phrase that would translate into the given mathematical expression.

109. $2z - 5$

110. $5x + 3$

111. $2(z - 5)$

112. $5(x + 3)$

113. $\frac{z + 3}{2}$

114. $\frac{t}{3} - 2t$

Explaining the Concepts

115. Explain the difference between a variable and a constant.

116. Explain the difference between a term and a factor.

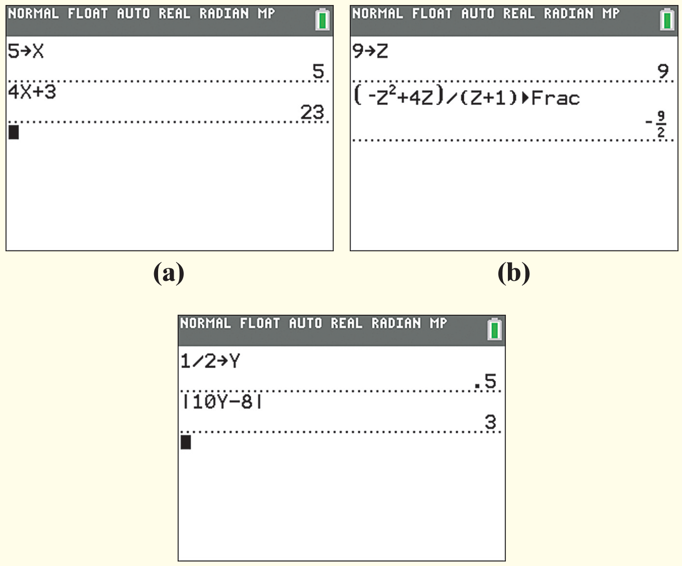
117. Explain what like terms are. Explain how the Distributive Property is used to combine like terms.

118. Explain the difference between the direction "evaluate the algebraic expression" and "simplify the algebraic expression."

Technology Exercises

Technology can be used to evaluate an algebraic expression. Figure 14 shows the results of Example 2 using a graphing calculator.

Figure 14



(a) (b) (c)

In Problems 119–126, use technology to evaluate each algebraic expression for the given values of the variable.

119. $-4x + 3$ for **(a)** $x = 0$ **(b)** $x = -3$ **(a)** 3 **(b)** 15

120. $-5x + 9$ for **(a)** $x = 4$ **(b)** $x = -3$ **(a)** -11 **(b)** 24

121. $4x^2 - 8x + 3$ for **(a)** $x = 5$ **(b)** $x = -2$ **(a)** 63 **(b)** 35

122. $-9x^2 + x - 5$ for **(a)** $x = 3$ **(b)** $x = -4$ **(a)** -83 **(b)** -153

123. $\frac{3z - 1}{z^2 + 1}$ for **(a)** $z = -2$ **(b)** $z = 8$ **(a)** $-\frac{7}{5}$ **(b)** $\frac{23}{65}$

124. $\frac{2y^2 + 5}{3y - 1}$ for **(a)** $y = 3$ **(b)** $y = -8$ **(a)** $\frac{23}{8}$ **(b)** $-\frac{133}{25}$

125. $|-9x + 5|$ for **(a)** $x = 8$ **(b)** $x = -3$ **(a)** 67 **(b)** 32

126. $|-3x^2 + 5x - 2|$ for **(a)** $x = 6$ **(b)** $x = -4$ **(a)** 80 **(b)** 70

127. Use technology to evaluate $\frac{2x + 1}{x - 5}$ when $x = 5$. What result is displayed? Why? Error; zero denominator

128. Use technology to evaluate $\frac{x + 2}{(x + 2)(x - 2)}$ when $x = -2$. What result is displayed? Why? Error; zero denominator

R.6 Square Roots

Objectives

1. Evaluate Square Roots of Perfect Squares
2. Determine Whether a Radical Is Rational, Irrational, or Not a Real Number
3. Evaluate Radical Expressions Containing Variables
4. Use the Product Rule to Simplify Square Roots

In Section R.4, exponents were introduced. Exponents indicate repeated multiplication. For example, 4^2 means $4 \cdot 4$, so $4^2 = 16$; $(-6)^2$ means $(-6)(-6) = 36$. This section will show how to "undo" the process of raising a number to the second power and ask questions such as, "What number, or numbers, when squared, give me 16?"

1 Evaluate Square Roots of Perfect Squares

A real number is squared when it is raised to the power of 2. The inverse of squaring a number is finding the **square root.** For example, since $5^2 = 25$ and $(-5)^2 = 25$, the square roots of 25 are -5 and 5. The square roots of $\frac{16}{49}$ are $-\frac{4}{7}$ and $\frac{4}{7}$.

The symbol $\sqrt{}$, called a **radical,** is used to denote the **principal square root,** or nonnegative (zero or positive) square root.

In Other Words

Taking the square root of a number "undoes" squaring a number.

Definition

If a is a nonnegative real number, the nonnegative real number b such that $b^2 = a$ is the **principal square root** of a and is denoted by $b = \sqrt{a}$.

In Other Words

The notation $b = \sqrt{a}$ means "produce the number b greater than or equal to 0 whose square is a."

For example, the principal square root of 25 is written $\sqrt{25} = 5$. We read $\sqrt{25} = 5$ as "the principal (or positive) square root of 25 is 5." If we want the negative square root of 25, use the expression $-\sqrt{25} = -5$.

Properties of Square Roots

- Every positive real number has two square roots, one positive and one negative.
- The square root of 0 is 0. That is, $\sqrt{0} = 0$.
- Use the symbol $\sqrt{}$, called a **radical**, to denote the nonnegative square root of a real number. The nonnegative square root is called the principal square root.
- The number under the radical sign is called the **radicand**. For example, the radicand in $\sqrt{25}$ is 25.
- For any non-negative real number c $(c \geq 0)$,

$$(\sqrt{c})^2 = c.$$

For example, $(\sqrt{4})^2 = 4$ and $(\sqrt{8.3})^2 = 8.3$.

To **evaluate** a principal square root, ask, "What is the nonnegative number whose square is equal to the radicand?"

EXAMPLE 1 Evaluating Principal Square Roots

Evaluate each principal square root.

(a) $\sqrt{36}$ **(b)** $\sqrt{\frac{1}{9}}$ **(c)** $\sqrt{0.01}$ **(d)** $(\sqrt{2.3})^2$

Solution

(a) Is there a positive number whose square is 36? Because $6^2 = 36$, $\sqrt{36} = 6$.

(b) $\sqrt{\frac{1}{9}} = \frac{1}{3}$ because $\left(\frac{1}{3}\right)^2 = \frac{1}{9}$.

(c) $\sqrt{0.01} = 0.1$ because $0.1^2 = 0.01$.

(d) $(\sqrt{2.3})^2 = 2.3$ because $(\sqrt{c})^2 = c$ when $c \geq 0$.

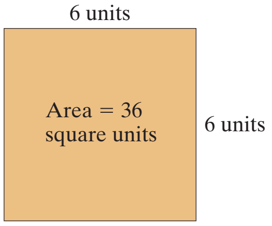
Classroom Example 1
Evaluate each principal square root.

(a) $\sqrt{49}$ (b) $\sqrt{\frac{1}{81}}$

(c) $\sqrt{0.09}$ (d) $(\sqrt{1.8})^2$

Answer: (a) 7 (b) $\frac{1}{9}$
(c) 0.3 (d) 1.8

Figure 15



A rational number is a **perfect square** if its principal square root is a rational number. Examples 1(a), (b), and (c) show that 36, $\frac{1}{9}$, and 0.01 are perfect squares. Perfect squares can be thought of geometrically as shown in Figure 15, where there is a square whose area is 36 square units. The square root of the area, $\sqrt{36}$, gives the length of each side of the square, 6 units.

Quick ✓

1. The symbol $\sqrt{\ }$ is called a radical sign.
2. If a is a nonnegative real number, the nonnegative number b such that $b^2 = a$ is the principal square root of a and is denoted by $b = \sqrt{a}$.
3. The square roots of 16 are -4 and 4.

In Problems 4–8, evaluate each principal square root.

4. $\sqrt{81}$ 9 **5.** $\sqrt{900}$ 30 **6.** $\sqrt{\frac{9}{4}}$ $\frac{3}{2}$ **7.** $\sqrt{0.16}$ 0.4 **8.** $(\sqrt{13})^2$ 13

EXAMPLE 2 Evaluating a Radical Expression

Classroom Example 2
Evaluate each expression.
(a) $-2\sqrt{25}$
(b) $\sqrt{144} + \sqrt{25}$
(c) $\sqrt{144 + 25}$
(d) $\sqrt{49 - 4 \cdot 3 \cdot 2}$
Answer: (a) -10 (b) 17 (c) 13 (d) 5

Evaluate each expression:

(a) $-4\sqrt{36}$ **(b)** $\sqrt{9} + \sqrt{16}$ **(c)** $\sqrt{9 + 16}$ **(d)** $\sqrt{64 - 4 \cdot 7 \cdot 1}$

Solution

(a) The expression $-4\sqrt{36}$ represents -4 times the positive square root of 36. To simplify, first find the positive (principal) square root of 36 and then multiply this result by -4.

$$\begin{aligned} -4\sqrt{36} &= -4 \cdot 6 \\ &= -24 \end{aligned}$$

(b) $$\begin{aligned} \sqrt{9} + \sqrt{16} &= 3 + 4 \\ &= 7 \end{aligned}$$

(c) $$\begin{aligned} \sqrt{9 + 16} &= \sqrt{25} \\ &= 5 \end{aligned}$$

(d) $$\begin{aligned} \sqrt{64 - 4 \cdot 7 \cdot 1} &= \sqrt{64 - 28} \\ &= \sqrt{36} \\ &= 6 \end{aligned}$$

Work Smart
In Examples 2(b) and (c), notice that
$$\sqrt{9} + \sqrt{16} \neq \sqrt{9 + 16}$$
In general,
$$\sqrt{a} + \sqrt{b} \neq \sqrt{a + b}$$
The radical acts as a grouping symbol, just like parentheses, so always simplify the radicand before taking the square root.

Quick ✓

In Problems 9–12, evaluate each expression.

9. $5\sqrt{9}$ 15 **10.** $\sqrt{36 + 64}$ 10 **11.** $\sqrt{36} + \sqrt{64}$ 14 **12.** $\sqrt{25 - 4 \cdot 3 \cdot (-2)}$ 7

2 Determine Whether a Radical Is Rational, Irrational, or Not a Real Number

Work Smart
Recall from Section R.1 that a rational number is a number that can be written as a quotient of two integers.

Work Smart
The square roots of negative real numbers are not real numbers.

Not all radical expressions will simplify to a rational number. For example, because there is no rational number whose square is 5, $\sqrt{5}$ is not a rational number. In fact, $\sqrt{5}$ is an *irrational* number. Remember, an irrational number is a number that cannot be written as the quotient of two integers.

What about evaluating $\sqrt{-16}$? Because any real number squared is positive, there is no real number whose square is -16. **Therefore, negative real numbers do not have square roots that are real numbers.**

These points are summarized below.

More Properties of Principal Square Roots

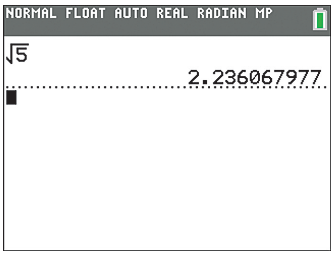
- The principal square root of a perfect square is a rational number.
- The principal square root of a positive rational number that is not a perfect square is an irrational number. For example, $\sqrt{20}$ is an irrational number because 20 is not a perfect square.
- The principal square root of a negative real number is not a real number. For example, $\sqrt{-2}$ is not a real number.

Classroom Example 3
Write $\sqrt{7}$ as a decimal rounded to two decimal places.
Answer: $\sqrt{7} \approx 2.65$

If a radical has a radicand that is not a perfect square, one of two things can be done:

1. Write a decimal approximation of the radical.
2. Simplify the radical using properties of radicals, if possible.

Figure 16



EXAMPLE 3 Writing a Radical as a Decimal Using Technology

Write $\sqrt{5}$ as a decimal rounded to two decimal places.

Solution

Figure 16 shows the result from a graphing calculator: $\sqrt{5} \approx 2.24$.

EXAMPLE 4 Determining Whether a Radical of an Integer Is Rational, Irrational, or Not a Real Number

Classroom Example 4
Follow the instructions for Example 4.
(a) $\sqrt{31}$ (b) $\sqrt{225}$ (c) $\sqrt{-41}$
Answer: (a) Irrational; 5.57 (b) Rational; 15 (c) Not a real number

Determine whether each radical is rational, irrational, or not a real number. Evaluate each real radical that is rational. For each radical that is irrational, express the radical as a decimal rounded to two decimal places.

(a) $\sqrt{51}$ **(b)** $\sqrt{169}$ **(c)** $\sqrt{-81}$

Solution

(a) $\sqrt{51}$ is irrational because 51 is not a perfect square. That is, there is no rational number whose square is 51. Using technology, $\sqrt{51} \approx 7.14$.

(b) $\sqrt{169}$ is a rational number because $13^2 = 169$, so $\sqrt{169} = 13$.

(c) $\sqrt{-81}$ is not a real number. There is no real number whose square is -81.

Work Smart
$\sqrt{-81}$ is not a real number, but $-\sqrt{81}$ is a real number because $-\sqrt{81} = -9$. Note the placement of the negative sign.

Quick ✓

13. *True or False* Negative numbers do not have square roots that are real numbers. True

In Problems 14–17, determine whether each radical is rational, irrational, or not a real number. Evaluate each radical that is rational. For each radical that is irrational, approximate the radical rounded to two decimal places.

14. $\sqrt{400}$ rational; 20 **15.** $\sqrt{40}$ irrational; ≈ 6.32 **16.** $\sqrt{-25}$ not real **17.** $-\sqrt{196}$ rational; -14

3 Evaluate Radical Expressions Containing Variables

What is $\sqrt{4^2}$? Because $4^2 = 16$, $\sqrt{4^2} = \sqrt{16} = 4$. This result suggests that $\sqrt{a^2} = a$ for any real number a. But wait. Does $\sqrt{(-4)^2} = -4$? No! $\sqrt{(-4)^2} = \sqrt{16} = 4$, so both $\sqrt{4^2}$ and $\sqrt{(-4)^2}$ equal 4. Regardless of whether the "a" in $\sqrt{a^2}$ is positive or negative, the result is positive. Therefore, $\sqrt{a^2} = a$ is incorrect, because the sign of a is unknown. How can this "formula" be fixed? Section R.2 demonstrated that $|a|$ is a positive number if a is nonzero. From this, the following results:

Teaching Tip
This is a hard concept for students to grasp. You may need to evaluate several numerical examples first.

In Other Words
The principal square root of a nonzero number squared will always be positive. The absolute value ensures this.

For any **real number** a,

$$\sqrt{a^2} = |a|$$

The bottom line is this—the square root of a variable expression raised to the second power is the absolute value of the variable expression.

EXAMPLE 5 Evaluating Radical Expressions Containing Variables

Classroom Example 5
Evaluate each expression.
(a) $\sqrt{5^2}$
(b) $\sqrt{(-17)^2}$
(c) $\sqrt{n^2}$
(d) $\sqrt{(5x-3)^2}$
Answer: (a) 5 (b) 17 (c) $|n|$
(d) $|5x-3|$

Evaluate each expression.

(a) $\sqrt{7^2}$ **(b)** $\sqrt{(-15)^2}$ **(c)** $\sqrt{x^2}$ **(d)** $\sqrt{(3x-1)^2}$

Solution

(a) $\sqrt{7^2} = 7$

(b) $\sqrt{(-15)^2} = |-15| = 15$

(c) It is unknown whether the real number x is positive, negative, or zero. To ensure that the result is positive or zero, write $\sqrt{x^2} = |x|$.

(d) $\sqrt{(3x-1)^2} = |3x-1|$

Quick ✓

18. $\sqrt{a^2} =$ $|a|$.

In Problems 19–22, evaluate each expression.

19. $\sqrt{10^2}$
10

20. $\sqrt{(-14)^2}$
14

21. $\sqrt{z^2}$
$|z|$

22. $\sqrt{(2x+3)^2}$
$|2x+3|$

4 Use the Product Rule to Simplify Square Roots

Recall that a number that is the square of a rational number is called a perfect square.

For example, $1^2 = 1$, $2^2 = 4$, $3^2 = 9$, $\left(\frac{2}{3}\right)^2 = \frac{4}{9}$ are all perfect squares.

Definition
A square root expression is **simplified** if the radicand does not contain any factors that are perfect squares.

Teaching Tip
Emphasize that $\sqrt{a}$ and $\sqrt{b}$ must be real numbers in order to apply the Product Rule.

For example, $\sqrt{18}$ is not simplified because 9 is a factor of 18, and 9 is a perfect square. But how do we simplify a square root expression? Consider the following:

$$\sqrt{25 \cdot 4} = \sqrt{100} = 10 \quad \text{and} \quad \sqrt{25} \cdot \sqrt{4} = 5 \cdot 2 = 10$$

Because both expressions equal 10, we might conclude that

$$\sqrt{25 \cdot 4} = \sqrt{25} \cdot \sqrt{4}$$

which suggests the following property of square roots:

In Other Words
$\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$ can be stated as follows: "The square root of a product equals the product of the square roots."

Product Rule for Square Roots
If $\sqrt{a}$ and $\sqrt{b}$ are nonnegative real numbers, then

$$\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$$

The Product Rule for Square Roots can be extended to any number of factors.

EXAMPLE 6 **How to Use the Product Property to Simplify Square Roots**

Simplify: $\sqrt{32}$

Classroom Example 6
Simplify: $\sqrt{12}$
Answer: $2\sqrt{3}$

Step-by-Step Solution

Step 1: Write the radicand as the product of factors, one of which is a perfect square.

Write 32 as $16 \cdot 2$ because 16 is a factor of 32, and 16 is a perfect square: $\sqrt{32} = \sqrt{16 \cdot 2}$

Step 2: Write the radicand as the product of radicals.

$\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$: $= \sqrt{16} \cdot \sqrt{2}$

Step 3: Take the square root of any perfect square.

$\sqrt{16} = 4$: $= 4 \cdot \sqrt{2}$

$= 4\sqrt{2}$ ●

Below is a summary of the steps used to simplify a square root expression.

Simplifying a Square Root Expression

Step 1: Write the radicand as the product of factors, one of which is a perfect square. Look for the largest factor of the radicand that is a perfect square.

Step 2: Use the Product Rule for Square Roots to write the radicand as the product of radicals.

Step 3: Take the square root of the perfect square.

EXAMPLE 7 **Using the Product Rule to Simplify Square Roots**

Simplify each of the following:

(a) $-3\sqrt{72}$ **(b)** $\sqrt{75x^2}$

Classroom Example 7
Simplify:
(a) $\sqrt{28}$ (b) $\sqrt{500}$
Answer: (a) $2\sqrt{7}$ (b) $10\sqrt{5}$

Solution

(a) The expression $-3\sqrt{72}$ is "negative three times the square root of seventy-two." Begin by simplifying $\sqrt{72}$. Because 36 is a factor of 2, and 36 is a perfect square, write 72 as $36 \cdot 2$.

$$-3\sqrt{72} = -3\sqrt{36 \cdot 2}$$

$\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$: $= -3\sqrt{36} \cdot \sqrt{2}$

$\sqrt{36} = 6$: $= -3 \cdot 6\sqrt{2}$

$= -18\sqrt{2}$

(b) $75 = 25 \cdot 3$

$$\sqrt{75x^2} = \sqrt{(25x^2) \cdot 3}$$

$\sqrt{ab} = \sqrt{a}\sqrt{b}$: $= \sqrt{25x^2} \cdot \sqrt{3}$

$\sqrt{ab} = \sqrt{a}\sqrt{b}$: $= \sqrt{25} \cdot \sqrt{x^2} \cdot \sqrt{3}$

$\sqrt{25} = 5$; $\sqrt{x^2} = |x|$: $= 5|x|\sqrt{3}$ ●

Sometimes there is more than one way to factor a number. For example, to simplify $\sqrt{48}$, 48 could also be factored as $48 = 4 \cdot 12$ or $16 \cdot 3$. Both eventually simplify to $4\sqrt{3}$.

$$\sqrt{48} = \sqrt{4 \cdot 12} = \sqrt{4} \cdot \sqrt{12} = 2\sqrt{3 \cdot 4} = 2 \cdot 2\sqrt{3} = 4\sqrt{3}$$
$$\sqrt{48} = \sqrt{16 \cdot 3} = \sqrt{16} \cdot \sqrt{3} \text{ or } 4\sqrt{3}$$

Notice that choosing the *largest* factor of a radicand that is a *perfect square* makes your mathematical life easier. Be sure to Work Smart!

Quick ✓

23. If $\sqrt{a}$ and $\sqrt{b}$ are nonnegative real numbers, then $\sqrt{ab} = \underline{\sqrt{a}\cdot\sqrt{b}}$.

24. To simplify $\sqrt{75}$, write it as $\sqrt{\underline{25}\cdot 3}$.

In Problems 25 and 26, simplify the square root.

25. $\sqrt{20}$ $2\sqrt{5}$ **26.** $\sqrt{72x^2}$ $6|x|\sqrt{2}$

Watch out! The direction "simplify" does not mean find a decimal approximation. The simplified form of a radical is its exact value, not its decimal approximation.

EXAMPLE 8 Simplifying an Expression Involving a Square Root

Classroom Example 8

Simplify: $\frac{6 - \sqrt{63}}{3}$

Answer: $2 - \sqrt{7}$

Simplify: $\frac{4 - \sqrt{20}}{2}$

Solution

To start, recognize that 4 is the largest perfect square factor of 20.

$$\frac{4-\sqrt{20}}{2} = \frac{4-\sqrt{4\cdot 5}}{2}$$

$\sqrt{a\cdot b} = \sqrt{a}\cdot\sqrt{b}$:
$$= \frac{4-\sqrt{4}\cdot\sqrt{5}}{2}$$
$$= \frac{4-2\sqrt{5}}{2}$$

Work Smart

$\frac{4 - \sqrt{20}}{2} \neq 2 - \sqrt{20}$

There are two options to finish simplifying the expression.

Factor out the 2 in the numerator: $= \frac{2(2-\sqrt{5})}{2}$

Divide out the common factor: $= 2 - \sqrt{5}$

$\frac{a-b}{c} = \frac{a}{c} - \frac{b}{c}$: $= \frac{4}{2} - \frac{2\sqrt{5}}{2}$

Divide out the common factors: $= 2 - \sqrt{5}$

Teaching Tip

Simplifying an expression such as the one in Example 8 is an important skill for students to have when they solve quadratic equations in Preparing for Chapter 2.

Quick ✓

In Problems 27 and 28, simplify the expression.

27. $\frac{6+\sqrt{45}}{3}$ $2+\sqrt{5}$ **28.** $\frac{-2+\sqrt{32}}{4}$ $\frac{-1+2\sqrt{2}}{2}$ or $-\frac{1}{2}+\sqrt{2}$

R.6 Exercises

MyLabMath® Underlined exercise numbers have complete video solutions in MyLabMath or may be accessed using the QR code to the right.

*Problems **1–28** are the **Quick ✓s** that follow the **EXAMPLES**.*

Building Skills

In Problems 29–38, evaluate each expression. See Objective 1.

29. $\sqrt{1}$ 1 **30.** $\sqrt{9}$ 3

<u>31</u>. $-\sqrt{100}$ -10 **32.** $-\sqrt{144}$ -12

<u>33</u>. $\sqrt{\frac{1}{4}}$ $\frac{1}{2}$ **34.** $\sqrt{\frac{4}{81}}$ $\frac{2}{9}$

35. $\sqrt{0.36}$ 0.6 **36.** $\sqrt{0.25}$ 0.5

37. $(\sqrt{1.6})^2$ 1.6 **38.** $(\sqrt{3.7})^2$ 3.7

In Problems 39–50, tell whether the radical is rational, irrational, or not a real number. If the radical is rational, find the exact value; if the radical is irrational, write the approximate value rounded to two decimal places. See Objective 2.

<u>39</u>. $\sqrt{-14}$ not a real number **40.** $\sqrt{-50}$ not a real number

41. $\sqrt{64}$ rational; 8 **42.** $\sqrt{121}$ rational; 11

<u>43</u>. $\sqrt{\frac{1}{16}}$ rational; $\frac{1}{4}$ **44.** $\sqrt{\frac{49}{100}}$ rational; $\frac{7}{10}$

45. $\sqrt{44}$ irrational; ≈ 6.63

46. $\sqrt{24}$ irrational; ≈ 4.90

47. $\sqrt{50}$ irrational; ≈ 7.07

48. $\sqrt{12}$ irrational; ≈ 3.46

49. $\sqrt{-16}$ not a real number

50. $\sqrt{-64}$ not a real number

In Problems 51–60, simplify each expression. See Objective 3.

51. $\sqrt{8^2}$ 8

52. $\sqrt{5^2}$ 5

53. $\sqrt{(-19)^2}$ 19

54. $\sqrt{(-13)^2}$ 13

55. $\sqrt{r^2}$ $|r|$

56. $\sqrt{w^2}$ $|w|$

57. $\sqrt{(x+4)^2}$ $|x+4|$

58. $\sqrt{(x-8)^2}$ $|x-8|$

59. $\sqrt{(4x-3)^2}$ $|4x-3|$

60. $\sqrt{(5x+2)^2}$ $|5x+2|$

In Problems 61–82, simplify each square root. See Objective 4.

61. $\sqrt{8}$ $2\sqrt{2}$

62. $\sqrt{27}$ $3\sqrt{3}$

63. $\sqrt{40}$ $2\sqrt{10}$

64. $\sqrt{32}$ $4\sqrt{2}$

65. $\sqrt{18}$ $3\sqrt{2}$

66. $\sqrt{52}$ $2\sqrt{13}$

67. $\sqrt{33}$ $\sqrt{33}$

68. $\sqrt{21}$ $\sqrt{21}$

69. $\sqrt{45}$ $3\sqrt{5}$

70. $\sqrt{50}$ $5\sqrt{2}$

71. $\sqrt{125}$ $5\sqrt{5}$

72. $\sqrt{200}$ $10\sqrt{2}$

73. $\sqrt{42}$ $\sqrt{42}$

74. $\sqrt{30}$ $\sqrt{30}$

75. $-2\sqrt{98}$ $-14\sqrt{2}$

76. $-4\sqrt{80}$ $-16\sqrt{5}$

77. $\sqrt{8^2-4(-2)(8)}$ $8\sqrt{2}$

78. $\sqrt{(-4)^2-4(-5)(2)}$ $2\sqrt{14}$

79. $\dfrac{4+\sqrt{36}}{2}$ 5

80. $\dfrac{5-\sqrt{100}}{5}$ -1

81. $\dfrac{9+\sqrt{18}}{3}$ $3+\sqrt{2}$

82. $\dfrac{10-\sqrt{75}}{5}$ $2-\sqrt{3}$

Mixed Practice

In Problems 83–112, simplify each expression.

83. $\sqrt{25+144}$ 13

84. $\sqrt{9+16}$ 5

85. $\sqrt{25}+\sqrt{144}$ 17

86. $\sqrt{9}+\sqrt{16}$ 7

87. $\sqrt{-144}$ not a real number

88. $\sqrt{-36}$ not a real number

89. $3\sqrt{25}$ 15

90. $-10\sqrt{16}$ -40

91. $5\sqrt{\dfrac{16}{25}}-\sqrt{144}$ -8

92. $2\sqrt{\dfrac{9}{4}}-\sqrt{4}$ 1

93. $\sqrt{8^2-4\cdot1\cdot7}$ 6

94. $\sqrt{9^2-4\cdot1\cdot20}$ 1

95. $\sqrt{(-5)^2-4\cdot2\cdot5}$ not a real number

96. $\sqrt{(-3)^2-4\cdot3\cdot2}$ not a real number

97. $\sqrt{63}$ $3\sqrt{7}$

98. $\sqrt{54}$ $3\sqrt{6}$

99. $\sqrt{144a^2}$ $12|a|$

100. $\sqrt{225b^2}$ $15|b|$

101. $\sqrt{27a^2}$ $3|a|\sqrt{3}$

102. $\sqrt{32x^2}$ $4|x|\sqrt{2}$

103. $\sqrt{4^2+6^2}$ $2\sqrt{13}$

104. $\sqrt{2^2+8^2}$ $2\sqrt{17}$

105. $\dfrac{-4-\sqrt{162}}{6}$ $\dfrac{-4-9\sqrt{2}}{6}$

106. $\dfrac{-6+\sqrt{48}}{8}$ $\dfrac{-3+2\sqrt{3}}{4}$

107. $\dfrac{7-\sqrt{98}}{14}$ $\dfrac{1-\sqrt{2}}{2}$

108. $\dfrac{-6+\sqrt{108}}{6}$ $-1+\sqrt{3}$

109. $\dfrac{-(-1)+\sqrt{(-1)^2-4\cdot6(-2)}}{2(-1)}$ -4

110. $\dfrac{-7+\sqrt{7^2-4\cdot2\cdot6}}{2\cdot2}$ $-\dfrac{3}{2}$

111. $\sqrt{(6-1)^2+(15-3)^2}$ 13

112. $\sqrt{(2-(-1))^2+(6-2)^2}$ 5

113. What are the square roots of 36? What is $\sqrt{36}$?
The square roots of 36 are 6 and -6; $\sqrt{36}=6$

114. What are the square roots of 64? What is $\sqrt{64}$?
The square roots of 64 are 8 and -8; $\sqrt{64}=8$

Math for the Future: Statistics *For Problems 115 and 116, use the formula* $Z=\dfrac{X-\mu}{\dfrac{\sigma}{\sqrt{n}}}$ *from statistics (a formula used to determine the value of one observation relative to that of another) to evaluate the expression for the given values. Write the exact value and then write your answer rounded to two decimal places.*

115. $X=120, \mu=100, \sigma=15, n=13$ $\dfrac{4\sqrt{13}}{3}$; 4.81

116. $X=40, \mu=50, \sigma=10, n=5$ $-\sqrt{5}$; -2.24

Explaining the Concepts

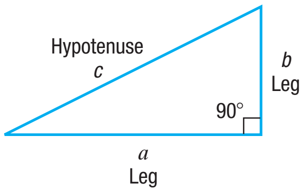
117. Explain why $\sqrt{a^2}=|a|$. Provide examples to support your explanation. Answers will vary.

R.7 Geometry Essentials

Objectives

1. Use the Pythagorean Theorem and Its Converse
2. Know Geometry Formulas
3. Understand Congruent Triangles and Similar Triangles

Figure 17
A right triangle



1 Use the Pythagorean Theorem and Its Converse

The *Pythagorean Theorem* is a statement about *right triangles*. A **right triangle** is one that contains a **right angle**—that is, an angle of 90°. The side of the triangle opposite the 90° angle is called the **hypotenuse;** the remaining two sides are called **legs.** In Figure 17, c represents the length of the hypotenuse, and a and b represent the lengths of the legs. Note the use of the symbol ⌐ to show the 90° angle. The Pythagorean Theorem is stated next.

Pythagorean Theorem

In a right triangle, the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the legs. That is, in the right triangle shown in Figure 17,

$$c^2 = a^2 + b^2 \qquad \textbf{(1)}$$

EXAMPLE 1

Finding the Hypotenuse of a Right Triangle

Classroom Example 1
Do Example 1 with $a = 6$ and $b = 8$.
Answer: 10

In a right triangle, one leg has length 4 and the other has length 3. What is the length of the hypotenuse?

Solution

Because the triangle is a right triangle, use the Pythagorean Theorem with $a = 4$ and $b = 3$ to find the length c of the hypotenuse. From equation (1),

$$c^2 = a^2 + b^2$$
$$c^2 = 4^2 + 3^2 = 16 + 9 = 25$$
$$c = \sqrt{25} = 5$$

Quick ✓

1. A(n) right triangle is one that contains an angle of 90 degrees. The longest side is called the hypotenuse.

In Problem 2, the lengths of the legs of a right triangle are given. Find the hypotenuse.

2. $a = 20$, $b = 21$ 29

The converse of the Pythagorean Theorem is also true.

Converse of the Pythagorean Theorem

In a triangle, if the square of the length of one side equals the sum of the squares of the lengths of the other two sides, the triangle is a right triangle. The 90° angle is opposite the longest side.

EXAMPLE 2

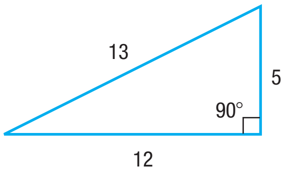
Verifying That a Triangle Is a Right Triangle

Classroom Example 2
Do Example 2 with lengths 7, 24, and 25.
Answer: $7^2 + 24^2 = 25^2$; 25

Show that a triangle whose sides are of lengths 5, 12, and 13 is a right triangle. Identify the hypotenuse.

(continued)

Figure 18



Solution

Square the lengths of the sides.

$$5^2 = 25 \qquad 12^2 = 144 \qquad 13^2 = 169$$

Notice that the sum of the first two squares (25 and 144) equals the third square (169). That is, because $5^2 + 12^2 = 13^2$, the triangle is a right triangle. The longest side, 13, is the hypotenuse. See Figure 18.

Quick ✓

3. ***True or False*** In a right triangle, the square of the length of the longest side equals the sum of the squares of the lengths of the other two sides. True

In Problems 4 and 5, the lengths of the sides of a triangle are given. Determine which are right triangles. For those that are, identify the hypotenuse.

4. 2, 3, 4 Not a right triangle **5.** 9, 40, 41 Right triangle; 41

EXAMPLE 3 Applying the Pythagorean Theorem

Classroom Example 3
Do Example 3.

The tallest building in the world is Burj Khalifa in Dubai, United Arab Emirates, at 2717 feet and 163 floors. The observation deck is 1483 feet above ground level. How far can a person standing on the observation deck see (with the aid of a telescope)? Use 3960 miles for the radius of Earth.

Source: Council on Tall Buildings and Urban Habitat (www.ctbuh.org)

Solution

From the center of Earth, draw two radii: one through Burj Khalifa and the other to the farthest point a person can see from the observation deck. See Figure 19. Apply the Pythagorean Theorem to the right triangle.

Since 1 mile = 5280 feet, 1483 feet $= \dfrac{1483}{5280}$ mile. Thus

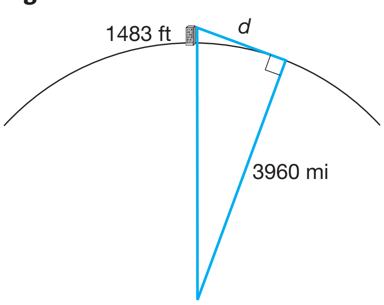
$$d^2 + (3960)^2 = \left(3960 + \frac{1483}{5280}\right)^2$$

$$d^2 = \left(3960 + \frac{1483}{5280}\right)^2 - (3960)^2 \approx 2224.58$$

Take the principal square root of 2224.58: $d \approx 47.17$

A person can see more than 47 miles from the observation deck.

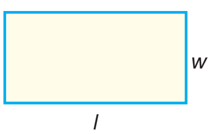
Figure 19



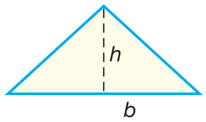
2 Know Geometry Formulas

Certain formulas from geometry are useful in solving algebra problems.

For a rectangle of length l and width w,

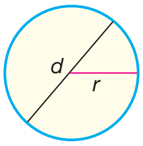


$$\text{Area} = lw \qquad \text{Perimeter} = 2l + 2w$$



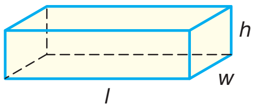
For a triangle with base b and altitude h,

$$\text{Area} = \frac{1}{2}bh$$



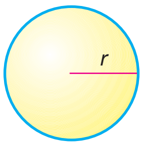
For a circle of radius r (diameter $d = 2r$ and $\pi \approx 3.14159$),

$$\text{Area} = \pi r^2 \qquad \text{Circumference} = 2\pi r = \pi d$$



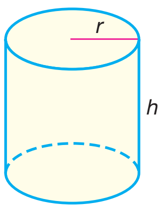
For a closed rectangular box of length l, width w, and height h,

$$\text{Volume} = lwh \qquad \text{Surface area} = 2lh + 2wh + 2lw$$



For a sphere of radius r,

$$\text{Volume} = \frac{4}{3}\pi r^3 \qquad \text{Surface area} = 4\pi r^2$$



For a closed right circular cylinder of height h and radius r,

$$\text{Volume} = \pi r^2 h \qquad \text{Surface area} = 2\pi r^2 + 2\pi rh$$

Quick ✓

6. The formula for the circumference C of a circle of radius r is $\underline{C = 2\pi r}$.

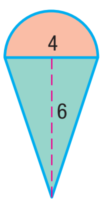
7. ***True or False*** The surface area of a sphere of radius r is $\frac{4}{3}\pi r^2$. False

EXAMPLE 4 Using Geometry Formulas

Classroom Example 4
Do Example 4 with the height of the triangle 8 cm and the base 5 cm.
Answer: 59.27 cm^2

A Christmas tree ornament is in the shape of a semicircle on top of a triangle. How many square centimeters (cm) of copper is required to make the ornament if the height of the triangle is 6 cm and the base is 4 cm?

Figure 20



Solution

See Figure 20. The amount of copper required equals the shaded area. This area is the sum of the areas of the triangle and the semicircle. The triangle has height $h = 6$ and base $b = 4$. The semicircle has diameter $d = 4$, so its radius is $r = 2$.

$$\begin{aligned} \text{Area} &= \text{Area of triangle} + \text{Area of semicircle} \\ &= \frac{1}{2}bh + \frac{1}{2}\pi r^2 = \frac{1}{2}(4 \text{ cm})(6 \text{ cm}) + \frac{1}{2}\pi \cdot (2 \text{ cm})^2 \qquad b = 4;\ h = 6;\ r = 2 \\ &= 12 \text{ cm}^2 + 2\pi \text{ cm}^2 \approx 18.28 \text{ cm}^2 \end{aligned}$$

About 18.28 cm^2 of copper is required.

Quick ✓

8. Find the area A and circumference C of a circle of radius 5 inches.
$A = 25\pi \text{ in}^2$; $C = 10\pi$ in

9. How many inches has a wheel with a diameter of 20 inches traveled after two revolutions? Round your answer to one decimal place. 125.7 in.

3 Understand Congruent Triangles and Similar Triangles

Throughout the text we will make reference to triangles, beginning here with a discussion of *congruent* triangles. According to thefreedictionary.com, the word **congruent** means "coinciding exactly when superimposed." For example, two angles are congruent if they have the same measure, and two line segments are congruent if they have the same length.

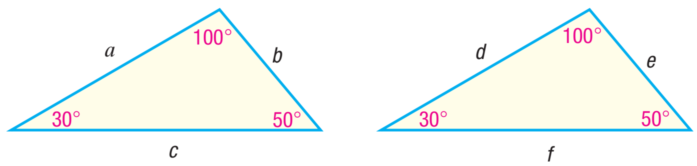
In Other Words
Two triangles are congruent if they have the same size and shape.

Definition

Two triangles are **congruent** if each pair of corresponding angles have the same measure and each pair of corresponding sides have the same length.

In Figure 21, corresponding angles are equal and the lengths of the corresponding sides are equal: $a = d$, $b = e$, and $c = f$, so the triangles are congruent.

Figure 21
Congruent Triangles

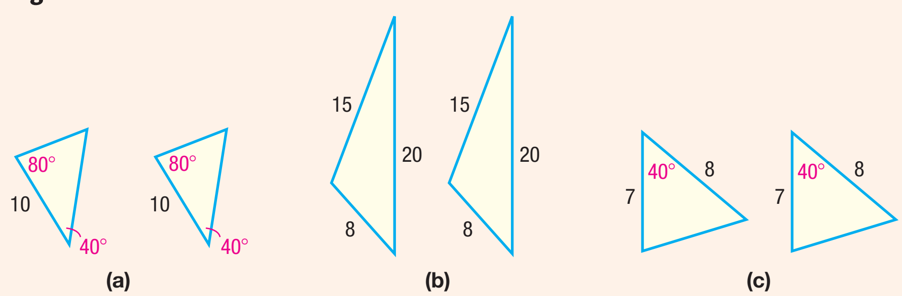


It is not necessary to verify that all three angles and all three sides are the same measure to determine whether two triangles are congruent.

Determining Congruent Triangles

1. **Angle–Side–Angle Case** Two triangles are congruent if the measures of two of the angles are equal and the lengths of the corresponding sides between the two angles are equal.
   For example, in Figure 22(a), the two triangles are congruent because the measures of two angles and the included side are equal.
2. **Side–Side–Side Case** Two triangles are congruent if the lengths of the corresponding sides of the triangles are equal.
   For example, in Figure 22(b), the two triangles are congruent because the lengths of the three corresponding sides are all equal.
3. **Side–Angle–Side Case** Two triangles are congruent if the lengths of two corresponding sides are equal and the measures of the angles between the two sides are the same.
   For example, in Figure 22(c), the two triangles are congruent because the lengths of two sides and the measure of the included angle are equal.

Figure 22



See the following definition to contrast congruent triangles with *similar* triangles.

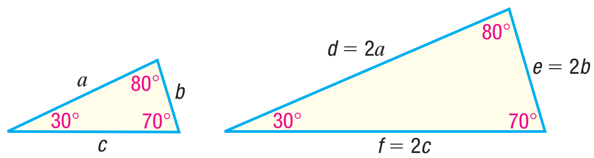
Definition

Two triangles are **similar** if the measures of the corresponding angles are equal and the lengths of the corresponding sides are proportional.

In Other Words
Two triangles are similar if they have the same shape, but (possibly) different sizes.

For example, the triangles in Figure 23 below are similar because the corresponding angles are equal. In addition, the lengths of the corresponding sides are proportional because each side in the triangle on the right is twice as long as each corresponding side in the triangle on the left. That is, the ratio of the corresponding sides is a constant: $\dfrac{d}{a} = \dfrac{e}{b} = \dfrac{f}{c} = 2.$

Figure 23
Similar Triangles

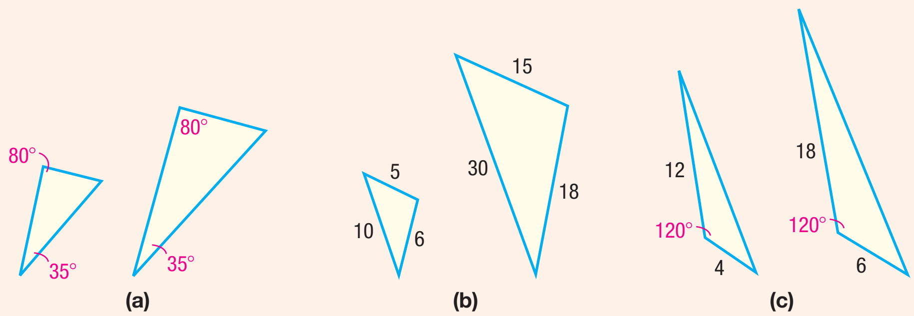


It is not necessary to verify that all three angles are equal and all three sides are proportional to determine whether two triangles are similar.

Determining Similar Triangles

1. **Angle–Angle Case** Two triangles are similar if two of the corresponding angles have equal measures.
   For example, in Figure 24(a), the two triangles are similar because two angles have equal measures.
2. **Side–Side–Side Case** Two triangles are similar if the lengths of all three sides of each triangle are proportional.
   For example, in Figure 24(b), the two triangles are similar because
   $$\frac{10}{30} = \frac{5}{15} = \frac{6}{18} = \frac{1}{3}$$
3. **Side–Angle–Side Case** Two triangles are similar if two corresponding sides are proportional and the angles between the two sides have equal measure.
   For example, in Figure 24(c), the two triangles are similar because $\dfrac{4}{6} = \dfrac{12}{18} = \dfrac{2}{3}$ and the angles between the sides have equal measure.

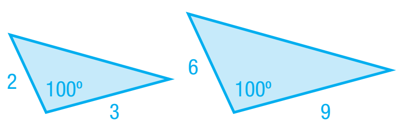
Figure 24



EXAMPLE 5

Using Similar Triangles

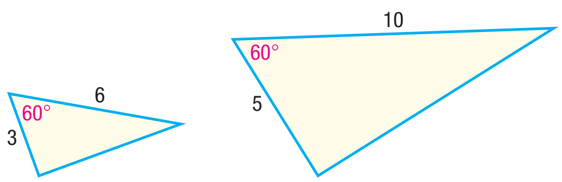
Classroom Example 5
Determain if the triangles are similar.



Answer: similar

Determine if the triangles in Figure 25 are similar.

Figure 25



(continued)

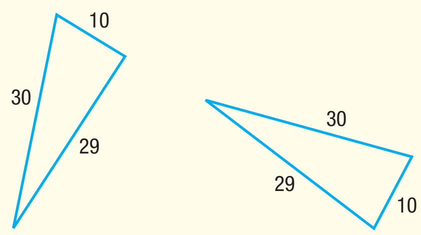
Solution

Because two corresponding sides and the angle between them are given, use the Side-Angle-Side case to determine if the two triangles are similar.

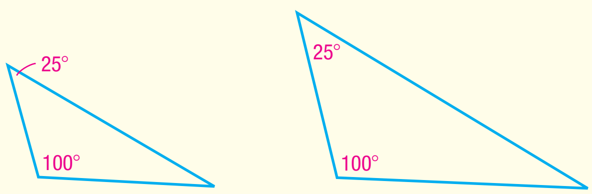
The angles between the two given sides are equal, 60°. Set up the ratios of the corresponding sides: $\frac{3}{5}, \frac{6}{10}$. Because $\frac{6}{10} = \frac{3}{5}$, the sides are proportional. The two triangles are similar because their corresponding sides are proportional and the angles between them have equal measure.

Quick ✓

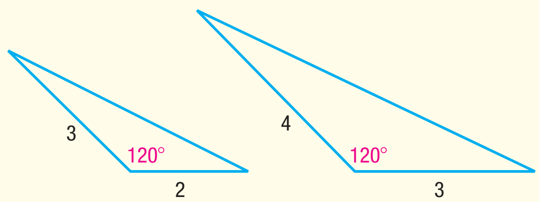
10. ***True or False*** The triangles shown are congruent. True



11. ***True or False*** The triangles shown are similar. True



12. ***True or False*** The triangles shown are similar. False



R.7 Exercises MyLabMath®

Problems **1–12** *are the* **Quick ✓s** *that follow the* **EXAMPLES**.

Building Skills

In Problems 13–18, the lengths of the legs of a right triangle are given. Find the hypotenuse. See Objective 1.

13. $a = 5, \quad b = 12$ 13

14. $a = 6, \quad b = 8$ 10

15. $a = 10, \quad b = 24$ 26

16. $a = 4, \quad b = 3$ 5

17. $a = 7, \quad b = 24$ 25

18. $a = 14, \quad b = 48$ 50

In Problems 19–26, the lengths of the sides of a triangle are given. Determine which are right triangles. For those that are, identify the hypotenuse. See Objective 1.

19. 3, 4, 5 Yes; 5

20. 6, 8, 10 Yes; 10

21. 4, 5, 6 No

22. 2, 2, 3 No

23. 7, 24, 25 Yes; 25

24. 10, 24, 26 Yes; 26

25. 6, 4, 3 No

26. 5, 4, 7 No

In Problems 27–38, find the area or volume. See Objective 2.

27. Find the area A of a rectangle with length 4 inches and width 2 inches. 8 in.^{2}

28. Find the area A of a rectangle with length 9 centimeters and width 4 centimeters. 36 cm^2

29. Find the area A of a triangle with height 4 inches and base 2 inches. 4 in.^{2}

30. Find the area A of a triangle with height 9 centimeters and base 4 centimeters. 18 cm^2

31. Find the area A and circumference C of a circle of radius 5 meters. $A = 25\pi$ m^2; $C = 10\pi$ m

32. Find the area A and circumference C of a circle of radius 2 feet. $A = 4\pi\ \text{ft}^2$; $C = 4\pi$ ft

33. Find the volume V and surface area S of a closed rectangular box with length 8 feet, width 4 feet, and height 7 feet. $V = 224\ \text{ft}^3$; $S = 232\ \text{ft}^2$

34. Find the volume V and surface area S of a closed rectangular box with length 9 inches, width 4 inches, and height 8 inches. $V = 288\ \text{in}^3$; $S = 280\ \text{in}^2$

35. Find the volume V and surface area S of a sphere of radius 4 centimeters. $V = \frac{256}{3}\pi\ \text{cm}^3$; $S = 64\pi\ \text{cm}^2$

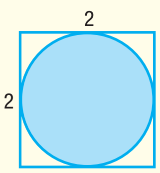
36. Find the volume V and surface area S of a sphere of radius 3 feet. $V = 36\pi\ \text{ft}^3$; $S = 36\pi\ \text{ft}^2$

37. Find the volume V and surface area S of a closed right circular cylinder with radius 9 inches and height 8 inches. $V = 648\pi\ \text{in}^3$; $S = 306\pi\ \text{in}^2$

38. Find the volume V and surface area S of a closed right circular cylinder with radius 8 inches and height 9 inches. $V = 576\pi\ \text{in}^3$; $272\pi\ \text{in}^2$

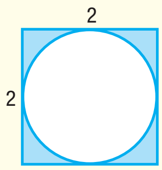
In Problems 39–42, find the area of the shaded region. See Objective 2.

39.



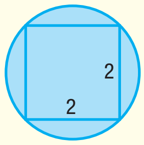
π square units

40.



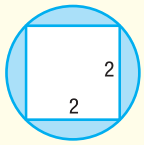
$4 - \pi$ square units

41.



2π square units

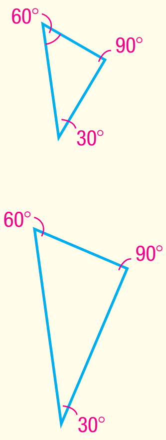
42.



$2\pi - 4$ square units

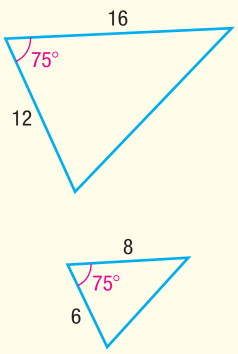
In Problems 43–46, determine if the triangles in each pair are similar. See Objective 3.

43.



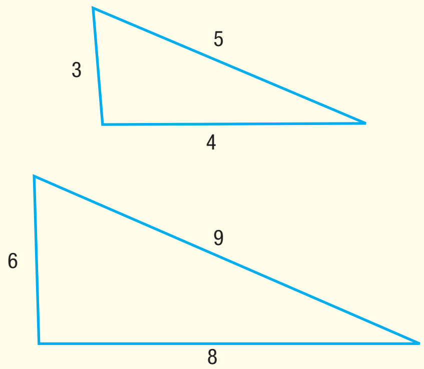
Similar

44.



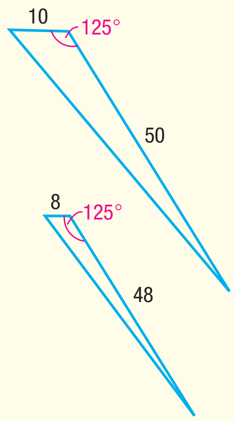
Similar

45.



Not Similar

46.



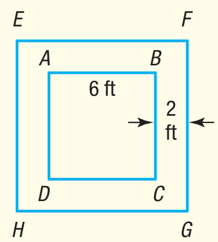
Not Similar

Applying the Concepts

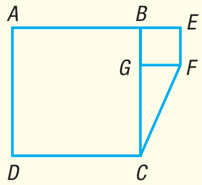
47. How many feet has a wheel with a diameter of 16 inches traveled after four revolutions? About 16.8 ft

48. How many revolutions will a circular disk with a diameter of 4 feet have completed after it has rolled 20 feet? ≈ 1.6 rev

49. In the figure shown, $ABCD$ is a square, with each side of length 6 feet. The width of the border (shaded portion) between the outer square $EFGH$ and $ABCD$ is 2 feet. Find the area of the border. $64\ \text{ft}^2$

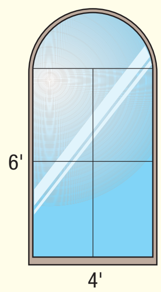


50. Refer to the figure. Square $ABCD$ has an area of 100 square feet; square $BEFG$ has an area of 16 square feet. What is the area of the triangle CGF? 12 ft^2



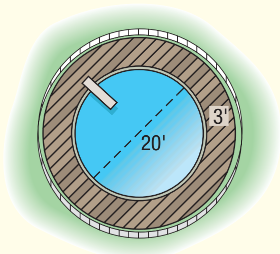
51. Architecture A **Norman window** consists of a rectangle surmounted by a semicircle. Find the area of the Norman window shown in the illustration. How much wood frame is needed to enclose the window?

$24 + 2\pi \approx 30.28 \text{ ft}^2$; $16 + 2\pi \approx 22.28$ ft



52. Construction A circular swimming pool that is 20 feet in diameter is enclosed by a wooden deck that is 3 feet wide. What is the area of the deck? How much fence is required to enclose the deck?

$69\pi \approx 216.77 \text{ ft}^2$; $26\pi \approx 81.68$ ft



In Problems 53–55, use the facts that the radius of Earth is 3960 miles and 1 mile = 5280 feet.

53. How Far Can You See? The conning tower of the U.S.S. *Silversides,* a World War II submarine now permanently stationed in Muskegon, Michigan, is approximately 20 feet above sea level. How far can you see from the conning tower? About 5.477 mi

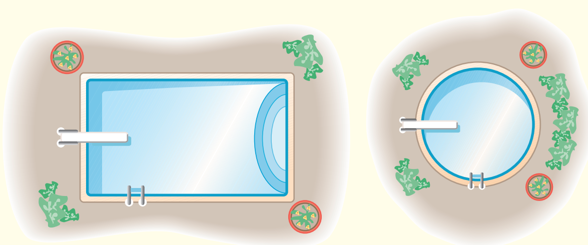
54. How Far Can You See? A person who is 6 feet tall is standing on the beach in Fort Lauderdale, Florida, and looks out onto the Atlantic Ocean. Suddenly, a ship appears on the horizon. How far is the ship from shore? About 3.0 mi

55. How Far Can You See? The deck of a destroyer is 100 feet above sea level. How far can a person see from the deck? How far can a person see from the bridge, which is 150 feet above sea level? 12.2 mi; 15.0 mi

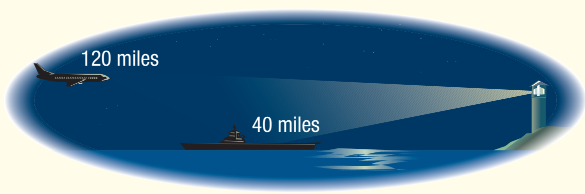
56. Suppose that m and n are positive integers with $m > n$. If $a = m^2 - n^2$, $b = 2mn$, and $c = m^2 + n^2$, show that a, b, and c are the lengths of the sides of a right triangle. (This formula can be used to find the sides of a right triangle that are integers, such as 3, 4, 5; 5, 12, 13; and so on. Such triplets of integers are called **Pythagorean triples**.) See Graphing Answer Section.

Explaining Concepts

57. You have 1000 feet of flexible pool siding and intend to construct a swimming pool. Experiment with rectangular-shaped pools with perimeters of 1000 feet. How do their areas vary? What is the shape of the rectangle with the largest area? Now compute the area enclosed by a circular pool with a perimeter (circumference) of 1000 feet. What would be your choice of shape for the pool? If rectangular, what is your preference for dimensions? Justify your choice. If your only objective is to have a pool that encloses the most area, what shape should you use? Answers will vary.



58. The Gibb's Hill Lighthouse, Southampton, Bermuda, in operation since 1846, stands 117 feet high on a hill 245 feet high, so its beam of light is 362 feet above sea level. A brochure states that the light itself can be seen on the horizon about 26 miles distant. Verify the accuracy of this information. The brochure further states that ships 40 miles away can see the light and that planes flying at 10,000 feet can see it 120 miles away. Verify the accuracy of these statements. What assumption did the brochure make about the height of the ship?



R.8 Laws of Exponents

Objectives

1. Simplify Exponential Expressions Using the Product Rule
2. Simplify Exponential Expressions Using the Quotient Rule
3. Evaluate Exponential Expressions with a Zero or Negative Exponent
4. Simplify Exponential Expressions Using the Power Rule
5. Simplify Exponential Expressions Containing Products or Quotients
6. Simplify Exponential Expressions Using the Laws of Exponents

Recall that if a is a real number and n is a positive integer, then the symbol a^n means that a should be used as a factor n times. That is,

$$a^n = \underbrace{a \cdot a \cdot \cdots \cdot a}_{n \text{ factors}}$$

For example,

$$4^3 = \underbrace{4 \cdot 4 \cdot 4}_{3 \text{ factors}}$$

In the notation a^n, a is called the **base** and n the **power** or **exponent.**

1 Simplify Exponential Expressions Using the Product Rule

Several general rules can be discovered for simplifying expressions involving *positive* integer exponents. The first rule is used when multiplying two exponential expressions that have the same base. Consider the following:

$$\underset{\text{Same base}}{x^2 \cdot x^4} = \underbrace{(x \cdot x)}_{2 \text{ factors}}\,\underbrace{(x \cdot x \cdot x \cdot x)}_{4 \text{ factors}} = \underbrace{x \cdot x \cdot x \cdot x \cdot x \cdot x}_{6 \text{ factors}} = \overset{\text{Sum of powers 2 and 4}}{\underset{\text{Same base}}{x^6}}$$

Work Smart

Read a^n as "a raised to the power of n" or "a raised to the nth power." We usually read a^2 as "a squared" and a^3 as "a cubed."

Based on the above, the following results:

In Other Words

When multiplying two exponential expressions with the same base, add the exponents. Then write the common base to the power of this sum.

Product Rule for Exponents (Positive Integer Exponents)

If a is a real number and m and n are positive integers, then

$$a^m a^n = a^{m+n}$$

EXAMPLE 1 Using the Product Rule to Simplify Expressions Involving Exponents

Classroom Example 1
Simplify each expression:
(a) $3^2 \cdot 3^3$ (b) $2z^2 \cdot 5z^4$

Answer: (a) $3^5 = 243$ (b) $10z^6$

Simplify each expression. All answers should contain only positive integer exponents.

(a) $2^2 \cdot 2^3$ **(b)** $3z^2 \cdot 4z^4$

Solution

(a)
$$2^2 \cdot 2^3 = 2^{2+3} = 2^5 = 32$$

(b)
$$3z^2 \cdot 4z^4 = 3 \cdot 4 \cdot z^2 \cdot z^4 = 12z^{2+4} = 12z^6$$

Work Smart

$5 = 5'$
$x = x'$

Quick ✓

1. In the notation a^n we call a the base and n the power or exponent.
2. If a is a real number, and m and n are positive integers, then $a^m a^n = $ a^{m+n}.

In Problems 3–7, simplify each expression. All answers should contain only positive integer exponents.

3. $5^2 \cdot 5$ 125
4. $(-3)^2(-3)^3$ -243
5. $y^4 y^3$ y^7
6. $(5x^2)(-2x^5)$ $-10x^7$
7. $6y^3(-y^2)$ $-6y^5$

2 Simplify Exponential Expressions Using the Quotient Rule

To find a general rule for the quotient of two exponential expressions with *positive* integer exponents, use the Reduction Property to divide out common factors. Consider the following:

$$\frac{y^6}{y^2} = \frac{\overbrace{y \cdot y \cdot y \cdot y \cdot \cancel{y} \cdot \cancel{y}}^{\text{6 factors}}}{\underbrace{\cancel{y} \cdot \cancel{y}}_{\text{2 factors}}} = \underbrace{y \cdot y \cdot y \cdot y}_{\text{4 factors}} = \overset{\substack{\text{Difference of powers 6 and 2} \\ \downarrow}}{y^4}$$

Conclude from this result that

$$\frac{y^6}{y^2} = y^{6-2} = y^4$$

This result is true in general.

In Other Words
When dividing two exponential expressions with a common base, subtract the exponent in the denominator from the exponent in the numerator. Then write the common base to the power of this difference.

Quotient Rule for Exponents (Positive Integer Exponents)

If a is a real number and m and n are positive integers, then

$$\frac{a^m}{a^n} = a^{m-n} \qquad \text{if } a \neq 0$$

EXAMPLE 2 **Using the Quotient Rule to Simplify Expressions Involving Exponents**

Classroom Example 2
Simplify each expression. Answers should contain only positive integer exponents:
(a) $\frac{6^4}{6}$ (b) $\frac{25m^8}{15m^3}$

Answer: (a) $6^3 = 216$ (b) $\frac{5}{3}m^5$

Simplify each expression. Answers should contain only positive integer exponents.

(a) $\frac{8^5}{8^3}$ **(b)** $\frac{27z^9}{12z^4}$

Solution

(a) $\frac{8^5}{8^3} = 8^{5-3}$
$= 8^2$
$= 64$

(b) $\frac{27z^9}{12z^4} = \frac{9 \cdot 3}{4 \cdot 3} z^{9-4}$
$= \frac{9}{4} z^5$

Quick ✓

8. *True or False* To divide two exponential expressions having the same base, keep the base and subtract the exponents. True

In Problems 9–12, simplify each expression. All answers should contain only positive integer exponents.

9. $\frac{5^6}{5^4}$ 25 **10.** $\frac{y^8}{y^6}$ y^2 **11.** $\frac{16a^6}{10a^5}$ $\frac{8}{5}a$ **12.** $\frac{-24b^5}{16b^3}$ $-\frac{3}{2}b^2$

3 Evaluate Exponential Expressions with a Zero or Negative Exponent

Now extend the definition of exponential expressions to *all* integer exponents. That is, evaluate exponential expressions where the exponent can be a positive integer, zero, or a negative integer. Begin with raising a real number to the 0 power.

Zero-Exponent Rule

If a is a nonzero real number, then

$$a^0 = 1 \quad \text{if } a \neq 0$$

This rule is based on the Product Rule and the Identity Property of Multiplication. From the Product Rule for Exponents, it is known that

$$\begin{aligned} a^0 a^n &= a^{0+n} \\ &= a^n \\ &= 1 \cdot a^n \end{aligned}$$

From the Identity Property of Multiplication, this means that $a^0 = 1$.

Suppose we want to simplify $\frac{z^3}{z^5}$. If the Quotient Rule for Exponents is used,

$$\frac{z^3}{z^5} = z^{3-5} = z^{-2}$$

This expression could also be simplified using the Reduction Property.

$$\frac{z^3}{z^5} = \frac{\cancel{z} \cdot \cancel{z} \cdot \cancel{z}}{\cancel{z} \cdot \cancel{z} \cdot \cancel{z} \cdot z \cdot z} = \frac{1}{z^2}$$

This implies that $z^{-2} = \frac{1}{z^2}$, which suggests that a raised to a negative power is defined as follows:

Negative-Exponent Rule

If n is a positive integer and if a is a nonzero real number, then

$$a^{-n} = \frac{1}{a^n} \quad \text{or} \quad \frac{1}{a^{-n}} = a^n \quad \text{if } a \neq 0$$

EXAMPLE 3 Evaluating Exponential Expressions Containing Integer Exponents

Classroom Example 3
Simplify each expression. All exponents should be positive integers only.

(a) 2^{-3} (b) $\frac{1}{5^{-2}}$

(c) $7p^0$ (d) $5b^{-4}$

Answer: (a) $\frac{1}{8}$ (b) 25 (c) 7 (d) $\frac{5}{b^4}$

Simplify each expression. All exponents should be positive integers.

(a) 3^{-4} **(b)** $\frac{1}{3^{-2}}$ **(c)** $5x^0$ **(d)** $4x^{-5}$

Solution

(a) $3^{-4} = \frac{1}{3^4} = \frac{1}{81}$

(b) $\frac{1}{3^{-2}} = 3^2 = 9$

(c) $5x^0 = 5 \cdot 1 = 5$

(d) $4x^{-5} = \frac{4}{x^5}$

Work Smart

Remember, $3^{-4} = \frac{1}{3^4}$ (not -3^4). The negative applies only to the exponent. Similarly, $4x^{-5} = \frac{4}{x^5}\left(\text{not } \frac{1}{4x^5}\right)$. The negative exponent applies only to the base "x" not to the coefficient.

Quick ✓

13. $a^0 = \underline{1}$, provided $a \neq \underline{0}$. **14.** $a^{-n} = \underline{\frac{1}{a^n}}$, provided $a \neq \underline{0}$.

In Problems 15–20, simplify each expression. All exponents should be positive integers.

15. 5^{-3} $\frac{1}{125}$ **16.** $5z^{-7}$ $\frac{5}{z^7}$ **17.** $\frac{1}{x^{-4}}$ x^4 **18.** $\frac{5}{y^{-3}}$ $5y^3$ **19.** -4^0 -1 **20.** $(-10)^0$ 1

EXAMPLE 4 Evaluating Exponential Expressions Containing Integer Exponents

Classroom Example 4
Simplify each expression. All exponents should be positive integers.
(a) $\left(\frac{3}{2}\right)^{-2}$ (b) $\left(\frac{1}{3}\right)^{-4}$

Answer: (a) $\frac{4}{9}$ (b) 81

Simplify each expression. All exponents should be positive integers.

(a) $\left(\frac{2}{3}\right)^{-3}$ **(b)** $\left(\frac{1}{7}\right)^{-2}$

Solution

(a)
$$\left(\frac{2}{3}\right)^{-3} = \frac{1}{\left(\frac{2}{3}\right)^{3}} = \frac{1}{\frac{2}{3}\cdot\frac{2}{3}\cdot\frac{2}{3}} = \frac{1}{\frac{8}{27}} = \frac{27}{8}$$

(b)
$$\left(\frac{1}{7}\right)^{-2} = \frac{1}{\left(\frac{1}{7}\right)^{2}} = \frac{1}{\frac{1}{7}\cdot\frac{1}{7}} = \frac{1}{\frac{1}{49}} = 49$$

The following shortcut is based on the results of Example 4:

In Other Words
To evaluate $\left(\frac{a}{b}\right)^{-n}$, determine the reciprocal of the base and then raise it to the nth power.

If a and b are real numbers and n is an integer, then
$$\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^{n} \quad \text{if } a \neq 0, b \neq 0$$

Quick ✓

In Problems 21–24, simplify each expression. All exponents should be positive integers.

21. $\left(\frac{4}{3}\right)^{-2}$ $\frac{9}{16}$ **22.** $\left(-\frac{1}{4}\right)^{-3}$ -64 **23.** $\left(\frac{3}{x}\right)^{-2}$ $\frac{x^2}{9}$ **24.** $\frac{5}{2^{-2}}$ 20

Now that we have definitions for 0 as an exponent and negative exponents, the Product Rule and Quotient Rule for Exponents can be restated, assuming that the exponent is any integer (positive, negative, or zero).

Teaching Tip
To show why m, n, or $m + n$ cannot be 0 or negative when $a = 0$, ask students to simplify 0^{-3} or 0^0.

Product Rule for Exponents

If a is a real number and m and n are integers, then
$$a^m a^n = a^{m+n}$$
If m, n, or $m + n$ is 0 or negative, then a cannot be 0.

Quotient Rule for Exponents

If a is a real number and if m and n are integers, then
$$\frac{a^m}{a^n} = a^{m-n} \quad \text{if } a \neq 0$$

Notice that allowing the exponents to be any integer (not just any positive integer) requires restrictions on the value of the base.

EXAMPLE 5

Using the Product Rule to Simplify Expressions Containing Exponents

Classroom Example 5
Simplify each expression. All exponents should be positive integers.
(a) $(-2)^3(-2)^{-6}$
(b) $\frac{5}{3}z^{-3}\left(-\frac{9}{20}z^4\right)$

Answer: (a) $-\frac{1}{8}$ (b) $-\frac{3}{4}z$

Simplify each expression. All exponents should be positive integers.

(a) $(-3)^2(-3)^{-4}$ **(b)** $\frac{3}{4}y^5 \cdot \frac{20}{9}y^{-2}$

Solution

(a)
$$(-3)^2(-3)^{-4} = (-3)^{2+(-4)} = (-3)^{-2} = \frac{1}{(-3)^2} = \frac{1}{9}$$

(b)
$$\frac{3}{4}y^5 \cdot \frac{20}{9}y^{-2} = \frac{3}{4}\cdot\frac{20}{9}y^{5+(-2)} = \frac{5}{3}y^3$$

EXAMPLE 6

Using the Quotient Rule to Simplify Expressions Containing Exponents

Classroom Example 6
Simplify each expression. All exponents should be positive integers.
(a) $\frac{a^{-3}}{a^{-7}}$ (b) $\frac{14x^4y}{2xy^5}$

Answer: (a) a^4 (b) $\frac{7x^3}{y^4}$

Simplify each expression. All exponents should be positive integers.

(a) $\frac{w^{-2}}{w^{-5}}$ **(b)** $\frac{20x^3y}{4xy^4}$

Solution

(a)
$$\frac{w^{-2}}{w^{-5}} = w^{-2-(-5)} = w^{-2+5} = w^3$$

(b)
$$\frac{20x^3y}{4xy^4} = 5x^{3-1}y^{1-4} = 5x^2y^{-3}$$
$$a^{-n} = \frac{1}{a^n}: \quad = \frac{5x^2}{y^3}$$

Quick ✓

In Problems 25–30, simplify each expression. All exponents should be positive integers.

25. $6^3 \cdot 6^{-5}$ $\frac{1}{36}$

26. $\frac{10^{-3}}{10^{-5}}$ 100

27. $(4x^2y^3)(5xy^{-4})$ $\frac{20x^3}{y}$

28. $\left(\frac{3}{4}a^3b\right)\left(\frac{8}{9}a^{-2}b^3\right)$ $\frac{2}{3}ab^4$

29. $\frac{-24b^5}{16b^{-3}}$ $-\frac{3}{2}b^8$

30. $\frac{50s^2t}{15s^5t^{-4}}$ $\frac{10t^5}{3s^3}$

4 Simplify Exponential Expressions Using the Power Rule

Another law of exponents applies when an exponential expression containing a power is itself raised to a power.

$$(3^2)^4 = \underbrace{3^2\cdot3^2\cdot3^2\cdot3^2}_{\text{4 factors}} = \underbrace{\underbrace{(3\cdot3)}_{\text{2 factors}}\;\underbrace{(3\cdot3)}_{\text{2 factors}}\;\underbrace{(3\cdot3)}_{\text{2 factors}}\;\underbrace{(3\cdot3)}_{\text{2 factors}}}_{2\cdot4\,=\,8\text{ factors}} = 3^8$$

The following results:

In Other Words
If an exponential expression contains a power raised to a power, keep the base and multiply the powers.

Power Rule for Exponential Expressions

If a is a real number and m and n are integers, then

$$(a^m)^n = a^{mn}$$

If m or n is 0 or negative, then a must not be 0.

EXAMPLE 7

Using the Power Rule to Simplify Exponential Expressions

Classroom Example 7
Simplify each expression. All exponents should be positive integers.
(a) $(3^2)^4$ (b) $[(-5)^3]^4$ (c) $(7^2)^0$
(d) $(z^{-2})^{-4}$

Answer: (a) 3^8 (b) $(-5)^{12}$ (c) 1
(d) z^8

Simplify each expression. All exponents should be positive integers.

(a) $(y^3)^5$ **(b)** $[(-3)^3]^2$ **(c)** $(6^3)^0$

Solution

(a) $(y^3)^5 = y^{3\cdot 5} = y^{15}$

(b) $[(-3)^3]^2 = (-3)^{3\cdot 2} = (-3)^6 = 729$

(c) $(6^3)^0 = 6^{3\cdot 0} = 6^0 = 1$

Quick ✓

In Problems 31–36, simplify each expression. All exponents should be positive integers.

31. $(2^2)^3$ 64 **32.** $(5^8)^0$ 1 **33.** $[(-4)^3]^2$ 4096

34. $(a^3)^5$ a^{15} **35.** $(z^3)^{-6}$ $\frac{1}{z^{18}}$ **36.** $(s^{-3})^{-7}$ s^{21}

5 Simplify Exponential Expressions Containing Products or Quotients

There are two additional laws of exponents. The first deals with raising a product to a power, and the second deals with raising a quotient to a power. Consider the following product to a power:

$$\begin{aligned}(xy)^3 &= (xy)(xy)(xy)\\ &= (x\cdot x\cdot x)(y\cdot y\cdot y)\\ &= x^3y^3\end{aligned}$$

The following results:

Work Smart

Do not use this rule to try to simplify $(a+b)^2$ as a^2+b^2 or $(a+b)^3$ as a^3+b^3. For this rule to apply, the base must be the *product* of two numbers, not a sum.

Product-to-a-Power Rule

If a and b are real numbers and n is an integer, then

$$(ab)^n = a^nb^n$$

If n is 0 or negative, neither a nor b can be 0.

EXAMPLE 8

Using the Product-to-a-Power Rule to Simplify Exponential Expressions

Classroom Example 8
Simplify each expression. All exponents should be positive integers.
(a) $(2a)^4$ (b) $(5z^6)^{-2}$ (c) $(-4b^3)^{-2}$

Answer: (a) $16a^4$ (b) $\frac{1}{25z^{12}}$

(c) $\frac{1}{16b^6}$

Simplify each expression. All exponents should be positive integers.

(a) $(3z)^4$ **(b)** $(-5y^{-2})^{-3}$ **(c)** $(-4a^2)^{-2}$

Solution

(a) $(3z)^4 = 3^4z^4 = 81z^4$

(b) $$\begin{aligned}(-5y^{-2})^{-3} &= (-5)^{-3}(y^{-2})^{-3}\\ &= \frac{y^{-2(-3)}}{(-5)^3}\\ &= \frac{y^6}{-125}\\ \frac{a}{-b} = -\frac{a}{b}: \quad &= -\frac{y^6}{125}\end{aligned}$$

(c) $$\begin{aligned}(-4a^2)^{-2} &= \frac{1}{(-4a^2)^2}\\ &= \frac{1}{(-4)^2(a^2)^2}\\ &= \frac{1}{16a^4}\end{aligned}$$

Quick ✓

In Problems 37–40, simplify each expression. All exponents should be positive integers.

37. $(5y)^3$ $125y^3$ **38.** $(6y)^0$ 1 **39.** $(3x^2)^4$ $81x^8$ **40.** $(4a^3)^{-2}$ $\frac{1}{16a^6}$

Now look at a quotient raised to a power:

$$\left(\frac{2}{3}\right)^4 = \left(\frac{2}{3}\right)\left(\frac{2}{3}\right)\left(\frac{2}{3}\right)\left(\frac{2}{3}\right) = \frac{2^4}{3^4}$$

The following results:

Quotient-to-a-Power Rule

If a and b are real numbers and n is an integer, then

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n} \qquad \text{if } b \neq 0$$

If n is negative or 0, then a cannot be 0.

EXAMPLE 9

Using the Quotient-to-a-Power Rule to Simplify Exponential Expressions

Classroom Example 9

Simplify each expression. All exponents should be positive integers.

(a) $\left(\frac{z}{5}\right)^3$ (b) $\left(\frac{5b^3}{c^2}\right)^2$

Answer: (a) $\frac{z^3}{125}$ (b) $\frac{25b^6}{c^4}$

Simplify each expression. All exponents should be positive integers.

(a) $\left(\frac{w}{4}\right)^3$ **(b)** $\left(\frac{2x^2}{y^3}\right)^4$

Solution

(a) $\left(\frac{w}{4}\right)^3 = \frac{w^3}{4^3}$

$= \frac{w^3}{64}$

(b) $\left(\frac{2x^2}{y^3}\right)^4 = \frac{(2x^2)^4}{(y^3)^4}$

$(ab)^n = a^n \cdot b^n$: $= \frac{2^4(x^2)^4}{(y^3)^4}$

$(a^m)^n = a^{m \cdot n}$: $= \frac{16x^{2 \cdot 4}}{y^{3 \cdot 4}}$

$= \frac{16x^8}{y^{12}}$

Quick ✓

In Problems 41–44, simplify each expression. All exponents should be positive integers.

41. $\left(\frac{z}{3}\right)^4$ $\frac{z^4}{81}$ **42.** $\left(\frac{x}{2}\right)^{-5}$ $\frac{32}{x^5}$ **43.** $\left(\frac{x^2}{y^3}\right)^4$ $\frac{x^8}{y^{12}}$ **44.** $\left(\frac{3a^{-2}}{b^4}\right)^3$ $\frac{27}{a^6b^{12}}$

6 Simplify Exponential Expressions Using the Laws of Exponents

The Laws of Exponents are now summarized.

Work Smart: Study Skills

When learning new rules, it is helpful to write each rule on a notecard that you can study with regularly. For example, you could put a^0, $a \neq 0$ on one side of a notecard and its value of 1 on the other side. This will allow you to quiz yourself. There is also a free app called Quizlet that may be used for electronic flash cards.

The Laws of Exponents

If a and b are real numbers and if m and n are integers, then assuming the expression is defined,

Zero-Exponent Rule:	$a^0 = 1$	if $a \neq 0$
Negative-Exponent Rule:	$a^{-n} = \frac{1}{a^n}$	if $a \neq 0$
Product Rule:	$a^m a^n = a^{m+n}$	
Quotient Rule:	$\frac{a^m}{a^n} = a^{m-n}$	if $a \neq 0$

(continued)

The Laws of Exponents (*continued*)

Power Rule:	$(a^m)^n = a^{mn}$	
Product-to-a-Power Rule:	$(ab)^n = a^n b^n$	
Quotient-to-a-Power Rule:	$\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$	if $b \neq 0$
Quotient-to-a-Negative-Power Rule:	$\left(\dfrac{a}{b}\right)^{-n} = \left(\dfrac{b}{a}\right)^n$	if $a \neq 0, b \neq 0$

Now do some examples where one or more of the preceding rules are used.

Classroom Example 10
Simplify each expression. All exponents should be positive integers.
(a) $\dfrac{x^2y^{-3}}{(x^3y)^2}$ (b) $(6c^{-2}d)\left(\dfrac{3c^{-3}d^4}{2d}\right)^2$

Answer: (a) $\dfrac{1}{x^4y^5}$ (b) $\dfrac{27d^7}{2c^8}$

EXAMPLE 10 Using the Laws of Exponents

Simplify each expression. All exponents should be positive integers. None of the variables is zero.

(a) $\dfrac{x^3y^{-1}}{(x^2y)^3}$ (b) $\left(\dfrac{3xy}{x^2y^{-2}}\right)^2\left(\dfrac{9x^2y^{-3}}{x^3y^2}\right)^{-1}$

Solution

(a)

$$\begin{aligned}
\frac{x^3y^{-1}}{(x^2y)^3} &\overset{(ab)^n = a^nb^n}{=} \frac{x^3y^{-1}}{(x^2)^3y^3} \\
(a^m)^n = a^{mn}: \quad &= \frac{x^3y^{-1}}{x^6y^3} \\
\frac{a^m}{a^n} = a^{m-n}: \quad &= x^{3-6}y^{-1-3} \\
&= x^{-3}y^{-4} \\
a^{-n} = \frac{1}{a^n}: \quad &= \frac{1}{x^3y^4}
\end{aligned}$$

(b)

$$\begin{aligned}
\left(\frac{3xy}{x^2y^{-2}}\right)^2\left(\frac{9x^2y^{-3}}{x^3y^2}\right)^{-1} &\overset{\frac{a^m}{a^n} = a^{m-n}}{=} (3x^{1-2}y^{1-(-2)})^2(9x^{2-3}y^{-3-2})^{-1} \\
&= (3x^{-1}y^3)^2(9x^{-1}y^{-5})^{-1} \\
(a \cdot b)^n = a^n \cdot b^n: \quad &= 3^2(x^{-1})^2(y^3)^2(9^{-1})(x^{-1})^{-1}(y^{-5})^{-1} \\
(a^m)^n = a^{m \cdot n}: \quad &= 9x^{-2}y^6\left(\frac{1}{9}\right)xy^5 \\
a^m a^n = a^{m+n}: \quad &= x^{-2+1}y^{6+5} \\
a^{-n} = \frac{1}{a^n}: \quad &= x^{-1}y^{11} \\
&= \frac{y^{11}}{x}
\end{aligned}$$

Work Smart: Study Skills

Many different approaches may be taken to simplify exponential expressions. In Example 10(b), the Quotient-to-a-Power Rule could be used first and then simplified instead of the method shown. Try working a problem one way and then working it again a second way to see if you obtain the same answer and to see which method you prefer.

Quick ✓

In Problems 45 and 46, simplify each expression. All exponents should be positive integers.

45. $\dfrac{(3x^2y)^2}{12xy^{-2}}$ $\dfrac{3x^3y^4}{4}$ **46.** $\left(\dfrac{2x^2y^{-1}}{x^{-2}y^2}\right)^2\left(\dfrac{4x^3y^2}{xy^{-2}}\right)^{-1}$ $\dfrac{x^6}{y^{10}}$

R.8 Exercises

MyLabMath® Underlined exercise numbers have complete video solutions in MyLabMath or may be accessed using the QR code to the right.

Problems **1–46** *are the* **Quick ✓s** *that follow the* **EXAMPLES**.

Mixed Practice

In Problems 47–92, simplify each expression. All exponents should be positive integers.

47. -5^2 -25

48. 5^{-2} $\frac{1}{25}$

49. -5^{-2} $-\frac{1}{25}$

50. -5^0 -1

51. $-8^2 \cdot 8^{-2}$ -1

52. $\frac{8^7}{8^5} \cdot 8^{-2}$ 1

53. $\left(\frac{4}{9}\right)^{-2}$ $\frac{81}{16}$

54. $\left(\frac{3}{4}\right)^{-3}$ $\frac{64}{27}$

<u>55.</u> $(-3)^2(-3)^{-5}$ $-\frac{1}{27}$

56. $(-4)^{-5}(-4)^3$ $\frac{1}{16}$

<u>57.</u> $\frac{(-4)^2}{(-4)^{-1}}$ -64

58. $\frac{(-3)^3}{(-3)^{-2}}$ -243

59. $\frac{2^3 \cdot 3^{-2}}{2^{-2} \cdot 3^{-4}}$ 288

60. $\frac{3^{-2} \cdot 5^3}{3^2 \cdot 5}$ $\frac{25}{81}$

<u>61.</u> $(6x)^3(6x)^{-3}$ 1 (assuming $x \neq 0$)

62. $(5a^2)^5(5a^2)^{-5}$ 1 (assuming $a \neq 0$)

<u>63.</u> $(2s^{-2}t^4)(-5s^2t)$ $-10t^5$

64. $(6ab)(3a^3b^{-4})$ $\frac{18a^4}{b^3}$

65. $\left(\frac{1}{4}xy\right)(20xy^{-2})$ $\frac{5x^2}{y}$

66. $(3xy^3)\left(\frac{1}{9}x^2y\right)$ $\frac{1}{3}x^3y^4$

<u>67.</u> $\frac{36x^7y^3}{9x^5y^2}$ $4x^2y$

68. $\frac{25a^2b^3}{5ab^6}$ $\frac{5a}{b^3}$

69. $\frac{21a^2b}{14a^3b^{-2}}$ $\frac{3b^3}{2a}$

70. $\frac{25x^{-2}y}{10xy^3}$ $\frac{5}{2x^3y^2}$

<u>71.</u> $(x^{-2})^4$ $\frac{1}{x^8}$

72. $(z^2)^{-6}$ $\frac{1}{z^{12}}$

73. $(3x^2y)^3$ $27x^6y^3$

74. $(5a^2b^{-1})^2$ $\frac{25a^4}{b^2}$

75. $\left(\frac{z}{4}\right)^{-3}$ $\frac{64}{z^3}$

76. $\left(\frac{x}{y}\right)^{-8}$ $\frac{y^8}{x^8}$

77. $(3a^{-3})^{-2}$ $\frac{a^6}{9}$

78. $(2y^{-2})^{-4}$ $\frac{y^8}{16}$

79. $(-2a^2b^3)^{-4}$ $\frac{1}{16a^8b^{12}}$

80. $(-4a^{-2}b^2)^{-2}$ $\frac{a^4}{16b^4}$

81. $\frac{2^3 \cdot xy^{-2}}{12(x^2)^{-2}y}$ $\frac{2x^5}{3y^3}$

82. $\frac{3^2 \cdot x^{-3}(y^2)^3}{15x^2y^8}$ $\frac{3}{5x^5y^2}$

<u>83.</u> $\left(\frac{15a^2b^3}{3a^{-4}b^5}\right)^{-2}$ $\frac{b^4}{25a^{12}}$

84. $\left(\frac{15x^4y^7}{18x^{-3}y}\right)^{-1}$ $\frac{6}{5x^7y^6}$

85. $(4x^4y^{-2})^{-1} \cdot (2x^2y^{-1})^2$ 1

86. $(9a^2b^{-4})^{-1} \cdot (3ab^{-2})^2$ 1

87. $\frac{(-2)^2x^3(yz)^2}{-4xy^{-2}z}$ $-x^2y^4z$

88. $\frac{(-3)^3a^3(ab)^{-2}}{9ab^4}$ $\frac{-3}{b^6}$

89. $\frac{(3x^{-1}yz^2)^2}{(xy^{-2}z)^3}$ $\frac{9y^8z}{x^5}$

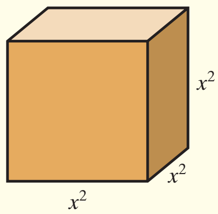
90. $\frac{(2ab^2c)^{-1}}{(a^{-1}b^3c^2)^{-2}}$ $\frac{b^4c^3}{2a^3}$

91. $\frac{(6a^3b^{-2})^{-1}}{(2a^{-2}b)^{-2}}\left(\frac{3ab^3}{2a^2b^{-3}}\right)^2$ $\frac{3b^{16}}{2a^9}$

92. $\left(\frac{a^{-3}b^{-1}}{2a^4b^{-2}}\right)^2\frac{(4a^2b)^2}{(2a^{-2}b)^3}$ $\frac{b}{2a^4}$

Applying the Concepts

△ **93. Cubes** Suppose the length of a side of a cube is x^2. Find the volume of the cube in terms of x.
x^6 cubic units



△ **94. Circles** The radius r of a circle is $\frac{d}{2}$, where d is the diameter. The area of a circle is given by the formula $A = \pi r^2$. Find the area of a circle in terms of its diameter d. $\frac{\pi d^2}{4}$ square units

In Problems 95–102, simplify each algebraic expression by rewriting each factor with a common base. (Hint: Consider that $8 = 2^3$.*)*

95. $\frac{2^{x+3}}{4}$ 2^{x+1}

96. $\frac{3^{2x}}{27}$ 3^{2x-3}

97. $3^x \cdot 27^{3x+1}$ 3^{10x+3}

98. $9^{-x} \cdot 3^{x+1}$ 3^{-x+1}

99. If $3^x = 5$, what does 3^{4x} equal? 625

100. If $4^x = 6$, what does 4^{5x} equal? 7776

101. If $2^x = 7$, what does 2^{-4x} equal? $\frac{1}{2401}$

102. If $5^x = 3$, what does 5^{-3x} equal? $\frac{1}{27}$

Explaining the Concepts

103. A friend of yours has a homework problem in which he must simplify $(x^4)^3$. He tells you that he thinks the answer is x^7. Is he right? If not, explain where he went wrong. **103–111.** See Graphing Answer Section.

104. A friend of yours is convinced that x^0 must equal 0. Write an explanation that details why $x^0 = 1$. Include any restrictions that must be placed on x.

105. Explain why a cannot be 0 when m, n, or $m + n$ is negative or 0 in the expression a^{m+n}. Use examples to support your explanation.

106. Explain why a cannot be 0 when n is negative or 0 in the expression $(a^m)^n$. Use examples to support your explanation.

107. Explain why neither a nor b can be 0 when n is 0 or negative in the expression $(ab)^n = a^n b^n$.

108. Provide a justification for the Product Rule for Exponents.

109. Provide a justification for the Quotient Rule for Exponents.

110. Provide a justification for the Power Rule for Exponents.

111. Provide a justification for the Product-to-a-Power Rule for Exponents.

R.9 Adding and Subtracting Polynomials

Objectives

1. Define Monomial and Determine the Coefficient and Degree of a Monomial
2. Define Polynomial and Determine the Degree of a Polynomial
3. Simplify Polynomials by Combining Like Terms

1 Define Monomial and Determine the Coefficient and Degree of a Monomial

Recall from Section R.5 that a *term* is a number or the product of a number and one or more variables raised to a power. The numerical factor of a term is the *coefficient*. For example, consider Table 4, where some algebraic expressions are given and their terms are identified.

Table 4

Algebraic Expression	Terms
$5x + 4$	$5x, 4$
$7x^2 - 8x + 3 = 7x^2 + (-8x) + 3$	$7x^2, -8x, 3$
$3x^2 + 7y^{-1}$	$3x^2, 7y^{-1}$

In this section, *polynomials* are studied. Polynomials have terms that are *monomials*.

Definition

A **monomial in one variable** is the product of a constant and a variable raised to a nonnegative integer power. A monomial in one variable is of the form

$$ax^k$$

where a is a constant, x is a variable, and $k \geq 0$ is an integer. The constant a is the **coefficient** of the monomial. If $a \neq 0$, then k is the **degree** of the monomial.

Work Smart

The nonnegative integers are 0, 1, 2, 3, ...

What if $a = 0$? Since $0x = 0x^2 = 0x^3 = \ldots, = 0$, a degree cannot be assigned to 0. Therefore, 0 has no degree.

EXAMPLE 1 Identifying the Coefficient and Degree of Monomials

Monomial	Coefficient	Degree
(a) $5x^3$	5	3
(b) $-\frac{2}{3}x^6$	$-\frac{2}{3}$	6
(c) $8 = 8x^0$	8	0
(d) $x^2 = 1x^2$	1	2
(e) $-x = -1 \cdot x$	-1	1
(f) 0	0	no degree

Now look at some expressions that are not monomials.

Classroom Example 1
Identify the coefficient and degree of each monomial:
(a) $2x^4$ (b) $-\frac{7}{4}n^6$ (c) 7
(d) x^3 (e) $-z$

Answer: (a) 2; 4 (b) $-\frac{7}{4}$; 6
(c) 7; 0 (d) 1; 3 (e) -1; 1

EXAMPLE 2 Expressions That Are Not Monomials

(a) $4x^{\frac{1}{2}}$ is not a monomial because the exponent of the variable x is $\frac{1}{2}$, and $\frac{1}{2}$ is not a nonnegative integer.

(b) $5x^{-3}$ is not a monomial because the exponent of the variable x is -3, and -3 is not a nonnegative integer.

Classroom Example 2
Examples of expressions that are not monomials:
(a) $5a^{-1}$ (b) $3x^{-4}$

Quick ✓

1. A monomial in one variable is the product of a number and a variable raised to a nonnegative integer power.

In Problems 2–5, determine whether the expression is a monomial. For those that are monomials, name the coefficient and give the degree.

2. $8x^5$ Monomial; 8; 5
3. $5x^{-2}$ Not a monomial
4. 12 Monomial; 12; 0
5. $x^{\frac{1}{3}}$ Not a monomial

A monomial may contain more than one variable factor. For example, the monomial $ax^m y^n$ has two variables, x and y, where m and n are nonnegative integers. The **degree of the monomial** $ax^m y^n$ is the sum of the exponents, $m + n$.

Classroom Example 3
Identify the degree and the coefficient of these monomials in more than one variable:
(a) $2a^3b^2$ (b) $-5xy^3$
Answer:
(a) Degree is 5; coefficient is 2
(b) Degree is 4; coefficient is −5

EXAMPLE 3 Monomials in More than One Variable

(a) $-4x^3y^4$ is a monomial in x and y of degree $3 + 4 = 7$. The coefficient is -4.

(b) $10ab^5$ is a monomial in a and b of degree $1 + 5 = 6$. The coefficient is 10.

Quick ✓

6. The degree of a monomial in the form $ax^m y^n$ is $m + n$.

In Problems 7–10, determine whether the expression is a monomial. For those that are monomials, determine the coefficient and degree.

7. $3x^5y^2$ Monomial; 3; 7
8. $-2m^3n$ Monomial; −2; 4
9. $4ab^{\frac{1}{2}}$ Not a monomial
10. $-xy$ Monomial; −1; 2

2 Define Polynomial and Determine the Degree of a Polynomial

We begin with a definition.

Definition
A **polynomial** is a monomial or the sum of monomials.

A polynomial is in **standard form** if it is written with the terms in descending order according to degree. The **degree of a polynomial** is the highest degree of all the terms of the polynomial. Remember, the degree of a nonzero constant is 0, and the number 0 has no degree.

Classroom Example 4
Examples of polynomials and their degree:
(a) $7a^3 - 4a^2 + 6a - 2$; degree $= 3$
(b) $3 - 11x + x^2$; degree $= 2$
(c) $2m^3n^2 - mn^3 + 4m^2n$; degree $= 5$
(d) $-x^3y + 2x^4y^2 + 1$; degree $= 6$
(e) 8; degree $= 0$
(f) 0; no degree

EXAMPLE 4 Examples of Polynomials

Polynomial	Degree
(a) $7x^3 - 2x^2 + 6x + 4$	3
(b) $3 - 8x + x^2 = x^2 - 8x + 3$	2
(c) $-7x^4 + 24$	4
(d) $x^3y^4 - 3x^3y^2 + 2x^3y$	7
(e) $p^2q - 8p^3q^2 + 3 = -8p^3q^2 + p^2q + 3$	5
(f) 6	0
(g) 0	No Degree

Classroom Example 5
Algebraic expressions that are not polynomials:
(a) $3x^{-2} - 6x + 2$ (b) $\frac{3}{x^2}$
(c) $\frac{2ab - 3}{ab + 4}$

EXAMPLE 5 Is the Algebraic Expression a Polynomial?

(a) $4x^{-2} - 5x + 1$ is not a polynomial because the exponent on the first term, -2, is negative.

(b) $\frac{4}{x^3}$ is not a polynomial because it can be written as $4x^{-3}$, and -3 is less than 0. Remember, the exponents on polynomials must be integers greater than or equal to 0.

(c) $\dfrac{8x^2 + 16}{2}$ is a polynomial of degree 2 because it can be written as $4x^2 + 8$ after dividing 2 into each term in the numerator.

(d) $\dfrac{3x^2 + 1}{x - 2}$ is not a polynomial because it is the quotient of two polynomials, and the expression cannot be simplified to a polynomial.

Quick ✓

11. *True or False* The degree of the polynomial $2m^2n + 5mn^3 - \frac{2}{3}m^2n^3$ is 5. True

In Problems 12–16, determine whether the algebraic expression is a polynomial. For those that are polynomials, determine the degree.

12. $-3x^3 + 7x^2 - x + 5$ Polynomial; 3

13. $5z^{-1} + 3$ Not a polynomial

14. $\dfrac{x - 1}{x + 1}$ Not a polynomial

15. $\dfrac{3x^2 - 9x + 27}{3}$ Polynomial; 2

16. $5p^3 - 8p^2 + p$ Polynomial; 3

Work Smart

The prefix "bi" means "two," as in bicycle. The prefix "tri" means "three," as in tricycle. The prefix "poly" means "many."

Certain polynomials have special names. A polynomial with exactly one term is a monomial; a polynomial that contains two monomials that are not like terms is called a **binomial;** and a polynomial that contains three monomials that are not like terms is called a **trinomial.** So

$-14x$ is a polynomial	but more specifically	$-14x$ is a monomial
$2x^3 - 5x$ is a polynomial	but more specifically	$2x^3 - 5x$ is a binomial
$-x^3 - 4x + 11$ is a polynomial	but more specifically	$-x^3 - 4x + 11$ is a trinomial
$3x^2 + 6x - 2$ is a polynomial	but more specifically	$3x^2 + 6x - 2$ is a trinomial

Work Smart

Remember, like terms have the same variable and the same exponent on the variable.

3 Simplify Polynomials by Combining Like Terms

To simplify a polynomial means to perform all indicated operations such as addition, and combine like terms. **To add polynomials, combine like terms.**

EXAMPLE 6

Simplifying Polynomials: Addition

Classroom Example 6
Simplify by finding the sum:
$(-5x^3 + 6x^2 + 2x - 7) + (3x^3 + 4x + 1)$

Answer:
$-2x^3 + 6x^2 + 6x - 6$

Simplify: $(-4x^3 + 9x^2 + x - 3) + (2x^3 + 6x + 5)$

Solution

The sum can be found using horizontal or vertical addition.

Horizontal Addition: The idea here is to combine like terms. The first step is to remove the parentheses.

$$\begin{aligned} (-4x^3 + 9x^2 + x - 3) + (2x^3 + 6x + 5) &= -4x^3 + 9x^2 + x - 3 + 2x^3 + 6x + 5 \\ \text{Rearrange terms:} &= -4x^3 + 2x^3 + 9x^2 + x + 6x - 3 + 5 \\ \text{Distributive Property:} &= (-4 + 2)x^3 + 9x^2 + (1 + 6)x + (-3 + 5) \\ \text{Simplify:} &= -2x^3 + 9x^2 + 7x + 2 \end{aligned}$$

Vertical Addition: Line up like terms in each polynomial vertically and then add the coefficients.

$$\begin{array}{rrrr} x^3 & x^2 & x^1 & x^0 \\ -4x^3 & +\,9x^2 & +\;\;x & -\,3 \\ 2x^3 & & +\,6x & +\,5 \\ \hline -2x^3 & +\,9x^2 & +\,7x & +\,2 \end{array}$$

Quick ✓

17. *True or False* $5y^3 + 7y^3 = 12y^6$ False

In Problems 18–20, simplify by adding the polynomials.

18. $(2x^2 - 3x + 1) + (4x^2 + 5x - 3)$ $6x^2 + 2x - 2$

19. $(5w^4 - 3w^3 + w - 8) + (-2w^4 + w^3 - 7w^2 + 3)$ $3w^4 - 2w^3 - 7w^2 + w - 5$

20. $\left(\frac{1}{2}x^2 - \frac{4}{3}x + 1\right) + \left(\frac{1}{4}x^2 + \frac{2}{3}x - 8\right)$ $\frac{3}{4}x^2 - \frac{2}{3}x - 7$

EXAMPLE 7 **Simplifying Polynomials in Two Variables: Addition**

Classroom Example 7
Find the sum:
$(5x^2y - 3xy + 12xy^2)$
$+ (x^2y + 12xy - 5xy^2)$
Answer: $6x^2y + 9xy + 7xy^2$

Simplify: $(5a^2b - 3ab + 2ab^2) + (a^2b + 5ab - 4ab^2)$

Solution

Although you may add horizontally or vertically, only the horizontal format is presented. The first step is to remove the parentheses.

$$\begin{aligned}
(5a^2b - 3ab + 2ab^2) + (a^2b + 5ab - 4ab^2) &= 5a^2b - 3ab + 2ab^2 + a^2b + 5ab - 4ab^2 \\
\text{Rearrange terms:} \quad &= 5a^2b + a^2b - 3ab + 5ab + 2ab^2 - 4ab^2 \\
\text{Distributive Property:} \quad &= (5+1)a^2b + (-3+5)ab + (2-4)ab^2 \\
\text{Simplify:} \quad &= 6a^2b + 2ab - 2ab^2
\end{aligned}$$

Quick ✓

In Problem 21, simplify by adding the polynomials.

21. $(8x^2y + 2x^2y^2 - 7xy^2) + (-3x^2y + 5x^2y^2 + 3xy^2)$ $5x^2y + 7x^2y^2 - 4xy^2$

Polynomials can be subtracted horizontally or vertically as well. Remember,

$$a - b = a + (-b)$$

Thus, to subtract one polynomial from another, add the opposite of *each term* in the polynomial following the subtraction sign and then combine like terms.

EXAMPLE 8 **Simplifying Polynomials: Subtraction**

Classroom Example 8
Find the difference:
$(6z^3 + 2z^2 - 5)$
$- (-3z^3 + 9z^2 - z + 1)$
Answer: $9z^3 - 7z^2 + z - 6$

Simplify: $(5z^3 + 3z^2 - 3) - (-2z^3 + 7z^2 - z + 2)$

Solution

Horizontal Subtraction: Recall that $a - b = a + (-1)b$.

$$\begin{aligned}
(5z^3 + 3z^2 - 3) - (-2z^3 + 7z^2 - z + 2) &= 5z^3 + 3z^2 - 3 + (-1)(-2z^3 + 7z^2 - z + 2) \\
\text{Distribute the } -1\text{:} \quad &= 5z^3 + 3z^2 - 3 + 2z^3 - 7z^2 + z - 2 \\
\text{Rearrange terms:} \quad &= 5z^3 + 2z^3 + 3z^2 - 7z^2 + z - 3 - 2 \\
\text{Combine like terms:} \quad &= 7z^3 - 4z^2 + z - 5
\end{aligned}$$

Vertical Subtraction: Line up like terms, change the sign of each coefficient of the second polynomial, and add.

$$\begin{array}{rrrr}
z^3 & z^2 & z^1 & z^0 \\
5z^3 & +\,3z^2 & & -\,3 \\
\underline{-(-2z^3} & \underline{+\,7z^2} & \underline{-\,z} & \underline{+\,2)}
\end{array}
\longrightarrow
\begin{array}{rrrr}
z^3 & z^2 & z^1 & z^0 \\
5z^3 & +\,3z^2 & & -\,3 \\
\underline{+\,2z^3} & \underline{-\,7z^2} & \underline{+\,z} & \underline{-\,2} \\
7z^3 & -\,4z^2 & +\,z & -\,5
\end{array}$$

Work Smart

Be careful with subtraction: The sign of *every term* of the second polynomial must be changed. Vertical subtraction will be used when we divide polynomials.

Quick ✓

In Problems 22–24, simplify by subtracting the polynomials.

22. $(5x^3 - 6x^2 + x + 9) - (4x^3 + 10x^2 - 6x + 7)$ $x^3 - 16x^2 + 7x + 2$

23. $(8y^3 - 5y^2 + 3y + 1) - (-3y^3 + 6y + 8)$ $11y^3 - 5y^2 - 3y - 7$

24. $(8x^2y + 2x^2y^2 - 7xy^2) - (-3x^2y + 5x^2y^2 + 3xy^2)$ $11x^2y - 3x^2y^2 - 10xy^2$

R.9 Exercises

MyLabMath® Underlined exercise numbers have complete video solutions in MyLabMath or may be accessed using the QR code to the right.

Problems **1–24** *are the* **Quick ✓s** *that follow the* **EXAMPLES**.

Building Skills

In Problems 25–32, determine the coefficient and degree of each monomial. See Objective 1.

25. $3x^2$ 3; 2

26. $5x^4$ 5; 4

27. $-8x^2y^3$ -8; 5

28. $-12xy$ -12; 2

29. $\frac{4}{3}x^6$ $\frac{4}{3}$; 6

30. $-\frac{5}{3}z^5$ $-\frac{5}{3}$; 5

31. 2 2; 0

32. -7 -7; 0

In Problems 33–36, state why each of the following is not a polynomial. See Objective 2. **33–36.** See Graphing Answer Section.

33. $2x^{-1} + 3x$

34. $6p^{-3} - p^{-2} + 3p^{-1}$

35. $\dfrac{4}{z-1}$

36. $\dfrac{x^2+2}{x}$

In Problems 37–52, determine whether the algebraic expression is a polynomial (Yes or No). If it is a polynomial, write the polynomial in standard form, determine the degree, and state whether it is a monomial, a binomial, or a trinomial. If it is a polynomial with more than three terms, say the expression is a polynomial. See Objective 2.

37. $5x^2 - 9x + 1$ Yes; $5x^2 - 9x + 1$; 2; trinomial

38. $-3y^2 + 8y + 1$ Yes; $-3y^2 + 8y + 1$; 2; trinomial

39. $\dfrac{-20}{n}$ No

40. $\dfrac{1}{x}$ No

41. $3y^{\frac{1}{3}} + 2$ No

42. $8m - 4m^{\frac{1}{2}}$ No

43. $\dfrac{5}{8}$ Yes; $\frac{5}{8}$; 0; monomial

44. -12 Yes; -12; 0; monomial

45. $5 - 8y + 2y^2$ Yes; $2y^2 - 8y + 5$; 2; trinomial

46. $7 - 5p + 2p^2 - p^3$ Yes; $-p^3 + 2p^2 - 5p + 7$; 3; polynomial

47. $7x^{-1} + 4$ No

48. $4y^{-2} + 6y - 1$ No

49. $3x^2y^2 + 2xy^4 + 4$ Yes; $2xy^4 + 3x^2y^2 + 4$; 5; trinomial

50. $4mn^3 - 2m^2n^3 + mn^8$

51. $4pqr + 2p^2q + 3pq^{\frac{1}{4}}$ No

52. $-2xyz^2 + 7x^3z - 8y^{\frac{1}{2}}z$ No

50. Yes; $mn^8 - 2m^2n^3 + 4mn^3$; 9; trinomial

In Problems 53–76, simplify each polynomial by adding or subtracting, as indicated. Express your answer as a single polynomial in standard form. See Objective 3.

53. $5z^3 + 8z^3$ $13z^3$

54 $10y^4 - 6y^4$ $4y^4$

55. $(x^2 + 5x + 1) + (3x^2 - 2x - 3)$ $4x^2 + 3x - 2$

56. $(x^2 - 4x + 1) + (5x^2 + 2x + 7)$ $6x^2 - 2x + 8$

57. $(6p^3 - p^2 + 3p - 4) + (2p^3 - 7p + 3)$ $8p^3 - p^2 - 4p - 1$

58. $(2w^3 - w^2 + 6w - 5) + (-3w^3 + 5w^2 + 9)$ $-w^3 + 4w^2 + 6w + 4$

59.
$$\begin{array}{r} 8x^3 + 4x^2 - 3x + 1 \\ + \; -x^3 - 2x^2 - 3x + 7 \\ \hline 7x^3 + 2x^2 - 6x + 8 \end{array}$$

60.
$$\begin{array}{r} -7n^3 + 2n^2 - 5n - 3 \\ + \; 2n^3 - 5n^2 + n + 1 \\ \hline -5n^3 - 3n^2 - 4n - 2 \end{array}$$

61. $(5x^2 + 9x + 4) - (3x^2 + 5x + 1)$ $2x^2 + 4x + 3$

62. $(7y^2 + 9y + 12) - (4y^2 + 8y - 3)$ $3y^2 + y + 15$

63. $(7s^2t^3 + st^2 - 5t - 8) - (4s^2t^3 + 5st^2 - 7)$ $3s^2t^3 - 4st^2 - 5t - 1$

64. $(-2x^3y^3 + 7xy - 3) - (x^3y^3 + 5y^2 + xy - 3)$ $-3x^3y^3 - 5y^2 + 6xy$

65. $(3 - 5x + x^2) + (-2 + 3x - 5x^2)$ $-4x^2 - 2x + 1$

66. $(-3 - 5z + 3z^2) + (1 + 2z + z^2)$ $4z^2 - 3z - 2$

67. $(6 - 2y + y^3) - (-2 + y^2 - 2y^3)$ $3y^3 - y^2 - 2y + 8$

68. $(8 - t^3) - (1 + 3t + 3t^2 + t^3)$ $-2t^3 - 3t^2 - 3t + 7$

69. $\left(\frac{1}{4}x^2 + \frac{3}{2}x + 3\right) + \left(\frac{1}{2}x^2 - \frac{1}{4}x - 2\right)$ $\frac{3}{4}x^2 + \frac{5}{4}x + 1$

70. $\left(\frac{3}{4}y^3 - \frac{1}{8}y + \frac{2}{3}\right) + \left(\frac{1}{2}y^3 + \frac{5}{12}y - \frac{5}{6}\right)$ $\frac{5}{4}y^3 + \frac{7}{24}y - \frac{1}{6}$

71. $(5x^2y^2 - 8x^2y + xy^2) + (3x^2y^2 + x^2y - 4xy^2)$ $8x^2y^2 - 7x^2y - 3xy^2$

72. $(7a^3b + 9ab^2 - 4a^2b) + (-4a^3b + 3a^2b - 8ab^2)$ $3a^3b - a^2b + ab^2$

73. $(3x^2y + 7xy^2 + xy) - (2x^2y - 4xy^2 - xy)$ $x^2y + 11xy^2 + 2xy$

74. $(-5xy^2 + 3xy - 9y^2) - (5xy^2 + 7xy - 8y^2)$
$-10xy^2 - 4xy - y^2$

75. $\begin{array}{r} 9a^3 + 2a^2 - 5a - 8 \\ -(5a^3 - 2a^2 + \ \ a - 6) \\ \hline 4a^3 + 4a^2 - 6a - 2 \end{array}$

76. $\begin{array}{r} -11p^3 + 8p^2 - p - 7 \\ -(-p^3 + 5p^2 + p + 1) \\ \hline -10p^3 + 3p^2 - 2p - 8 \end{array}$

Applying the Concepts

77. Add $5x^3 - 5x + 3$ to $-4x^3 + x^2 - 2x + 1$.
$x^3 + x^2 - 7x + 4$

78. Add $2x^3 - 3x^2 - 5x + 7$ to $x^3 + 3x^2 - 6x - 4$.
$3x^3 - 11x + 3$

79. Subtract $4b^3 - b^2 + 3b - 1$ from $2b^3 + 5b^2 - b + 3$.
$-2b^3 + 6b^2 - 4b + 4$

80. Subtract $2q^3 - 3q^2 + 7q - 2$ from $-5q^3 + q^2 + 2q - 1$. $-7q^3 + 4q^2 - 5q + 1$

In Problems 81–84, simplify each of the following.

81. $(2x^2 - 3x + 1) + (x^2 + 9) - (4x^2 - 2x - 5)$
$-x^2 - x + 15$

82. $(y^2 + 6y - 2) + (4y^2 - 9) - (2y^2 - 7y - 10)$
$3y^2 + 13y - 1$

83. $(4n - 3) + (n^3 - 9) - (2n^2 - 7n + 3)$
$n^3 - 2n^2 + 11n - 15$

84. $(y^3 - 1) + (y^2 - 9) - (y^3 + 2y^2 - 7y + 3)$
$-y^2 + 7y - 13$

Explaining the Concepts

85, 87–88. See Graphing Answer Section.

85. Explain the difference between a term and a monomial.

86. How many terms are in a binomial? How many terms are in a trinomial? 2; 3

87. Give a definition of *polynomial* using your own words. Provide examples of polynomials that are monomials, binomials, and trinomials.

88. Explain why the degree of the sum of two polynomials is at most the degree of the polynomial of highest degree.

R.10 Multiplying Polynomials

Objectives

1. Multiply a Monomial and a Polynomial
2. Multiply Two Binomials
3. Multiply Two Polynomials
4. Multiply Special Products

Multiplying polynomials is based on the Product Rule for Exponents (Section R.8) and the Distributive Property (Section R.2). The following example will help review these concepts.

EXAMPLE 1

Review the Product Rule for Exponents and the Distributive Property

Classroom Example 1
(a) Simplify: $(3xy^2)(-5x^2y^3)$
(b) Remove parentheses: $3(m-4)$
Answer: (a) $-15x^3y^5$ (b) $3m-12$

(a) Simplify: $(2a^3b)(-6a^2b^4)$

(b) Remove the parentheses: $2(z+3)$

Solution

(a)

$$\begin{aligned}(2a^3b)(-6a^2b^4) &= (2(-6))(a^3\cdot a^2)(b\cdot b^4)\\ \text{Product Rule: } a^m\cdot a^n = a^{m+n}\text{:}\quad &= -12a^{3+2}b^{1+4}\\ &= -12a^5b^5\end{aligned}$$

(b) The Distributive Property is used to remove parentheses:

$$\begin{aligned}2(z+3) &= 2\cdot z + 2\cdot 3\\ &= 2z+6\end{aligned}$$

Quick ✓

In Problems 1–4, simplify each expression completely. All exponents should be positive.

1. $(3x^5)(2x^2)$ $6x^7$

2. $(-7a^3b^2)(3ab^4)$ $-21a^4b^6$

3. $\left(\frac{2}{3}x^4\right)\left(\frac{15}{8}x\right)$ $\frac{5}{4}x^5$

4. $-3(x+2)$ $-3x-6$

1 Multiply a Monomial and a Polynomial

Example 1(b) shows that in multiplying a binomial by a monomial, the Distributive Property is used to remove the parentheses. In general, when multiplying a polynomial by a monomial, use the following property:

In Other Words
The Extended Form of the Distributive Property says to multiply each term in parentheses by a.

Extended Form of the Distributive Property

$$a(b_1+b_2+\cdots+b_n) = a\cdot b_1 + a\cdot b_2 + \cdots + a\cdot b_n$$

where $a, b_1, b_2, \ldots, b_n$ are real numbers.

EXAMPLE 2

Using the Extended Form of the Distributive Property

Classroom Example 2
Multiply and simplify each expression:
(a) $2x^2(x^2+3x+5)$
(b) $\frac{1}{2}yz^3\left(\frac{4}{3}yz^2+8y+\frac{1}{4}\right)$
Answer: (a) $2x^4+6x^3+10x^2$
(b) $\frac{2}{3}y^2z^5+4y^2z^3+\frac{1}{8}yz^3$

Multiply and simplify each of the following expressions:

(a) $3x^2(x^2+4x+2)$

(b) $\frac{1}{2}xy^3\left(\frac{2}{3}xy^2+\frac{6}{5}y+\frac{3}{4}\right)$

(continued)

Solution

(a)
$$3x^2(x^2 + 4x + 2) = 3x^2 \cdot x^2 + 3x^2 \cdot 4x + 3x^2 \cdot 2$$
$$= 3x^4 + 12x^3 + 6x^2$$

(b)
$$\frac{1}{2}xy^3\left(\frac{2}{3}xy^2 + \frac{6}{5}y + \frac{3}{4}\right) = \frac{1}{2}xy^3 \cdot \frac{2}{3}xy^2 + \frac{1}{2}xy^3 \cdot \frac{6}{5}y + \frac{1}{2}xy^3 \cdot \frac{3}{4}$$
$$= \frac{1}{3}x^2y^5 + \frac{3}{5}xy^4 + \frac{3}{8}xy^3$$

Quick ✓

In Problems 5–7, multiply and simplify each expression.

5. $5x(x^2 + 3x + 2)$ $5x^3 + 15x^2 + 10x$

6. $2xy(3x^2 - 5xy + 2y^2)$ $6x^3y - 10x^2y^2 + 4xy^3$

7. $\frac{3}{4}y^2\left(\frac{4}{3}y^2 + \frac{2}{9}y + \frac{16}{3}\right)$ $y^4 + \frac{1}{6}y^3 + 4y^2$

Teaching Tip
Advise students that they will be reversing the process later. Here, we change a product into a sum. Later, we change a sum into a product.

2 Multiply Two Binomials

To find the product of two binomials, distribute the first binomial to each term in the second binomial. Just as with addition and subtraction of polynomials, use either the horizontal or the vertical format.

EXAMPLE 3 **Multiplying Two Binomials**

Classroom Example 3
Find the product: $(2y - 3)(5y + 6)$
Answer: $10y^2 - 3y - 18$

Find the product: $(3x + 4)(2x - 5)$

Solution

VERTICAL MULTIPLICATION

$$\begin{array}{rrrl}
 & 3x & +\ 4 & \\
\times & 2x & -\ 5 & \\
\hline
 & -15x & -\ 20 & \longleftarrow -5(3x+4) \\
6x^2 & +\ 8x & & \longleftarrow 2x(3x+4) \\
\hline
6x^2 & -\ 7x & -\ 20 &
\end{array}$$

HORIZONTAL MULTIPLICATION

Distribute $3x + 4$ to each term in $2x - 5$.

$$(3x + 4)(2x - 5) = (3x + 4)(2x) + (3x + 4)(-5)$$

Distribute: $= 3x \cdot 2x + 4 \cdot 2x + 3x \cdot (-5) + 4 \cdot (-5)$

$= 6x^2 + 8x - 15x - 20$

Combine like terms: $= 6x^2 - 7x - 20$

In either case, $(3x + 4)(2x - 5) = 6x^2 - 7x - 20$.

The FOIL Method

To multiply two binomials, a method known as the **FOIL method** can also be used. The acronym FOIL stands for **F**irst, **O**uter, **I**nner, **L**ast and is used to help remember the order in which terms should be multiplied, as illustrated below.

First Last

$$(ax + b)(cx + d) = \overset{F}{ax \cdot cx} + \overset{O}{ax \cdot d} + \overset{I}{b \cdot cx} + \overset{L}{b \cdot d}$$

Inner
Outer

Work Smart
The FOIL method is a form of the Distributive Property and can be used only when two binomials are being multiplied.

EXAMPLE 4 Using the FOIL Method to Multiply Two Binomials

Classroom Example 4
Find the product:
(a) $(2a + 5)(3a - 4)$
(b) $(4y + z)(y - 2z)$
Answer: (a) $6a^2 + 7a - 20$
(b) $4y^2 - 7yz - 2z^2$

Find the product:

(a) $(4y - 3)(2y + 5)$

(b) $(2m + n)(m - 3n)$

Solution

First Last

Inner

Outer

$$\textbf{(a)}\quad (4y-3)(2y+5) = \overset{F}{4y\cdot 2y} + \overset{O}{4y\cdot 5} - \overset{I}{3\cdot 2y} - \overset{L}{3\cdot 5}$$
$$= 8y^2 + 20y - 6y - 15$$
$$= 8y^2 + 14y - 15$$

$$\textbf{(b)}\quad (2m+n)(m-3n) = \overset{F}{2m\cdot m} + \overset{O}{2m\cdot(-3n)} + \overset{I}{n\cdot m} + \overset{L}{n\cdot(-3n)}$$
$$= 2m^2 - 6mn + mn - 3n^2$$
$$= 2m^2 - 5mn - 3n^2$$

Quick ✓

8. The acronym FOIL stands for First, Outer, Inner, Last.

In Problems 9–11, find the product.

9. $(x + 4)(x + 1)$ $x^2 + 5x + 4$ **10.** $(3v + 5)(2v - 3)$ $6v^2 + v - 15$

11. $(2a - b)(a + 5b)$ $2a^2 + 9ab - 5b^2$

3 Multiply Two Polynomials

Work Smart
When multiplying polynomials, it is a good idea to write the polynomials in descending order of degree.

When multiplying a trinomial by a binomial or a trinomial by a trinomial, make repeated use of the Extended Form of the Distributive Property. The approach is similar to the one used in Example 3. It is a good idea to write each polynomial in standard form. Although either the horizontal or the vertical format may be used, only the horizontal format is presented.

EXAMPLE 5 Multiplying Two Polynomials

Classroom Example 5
Find the product:
(a) $(2x + 3)(x^2 + 5x - 1)$
(b) $(x^2 + 4x + 2)(2x^2 - x + 5)$
Answer: (a) $2x^3 + 13x^2 + 13x - 3$
(b) $2x^4 + 7x^3 + 5x^2 + 18x + 10$

Find the product:

(a) $(3x + 1)(x^2 + 4x - 3)$

(b) $(x^2 + 5x + 2)(2x^2 - x + 3)$

Solution

(a) Begin by distributing the binomial $3x + 1$ to each term in the trinomial.

$$(3x+1)(x^2+4x-3) = (3x+1)\cdot x^2 + (3x+1)\cdot 4x + (3x+1)\cdot(-3)$$

Distributive Property: $= 3x^3 + x^2 + 12x^2 + 4x - 9x - 3$

Combine like terms: $= 3x^3 + 13x^2 - 5x - 3$

(b) Distribute $x^2 + 5x + 2$ to each term in the second trinomial.

$$(x^2+5x+2)(2x^2-x+3) = (x^2+5x+2)\cdot 2x^2 + (x^2+5x+2)\cdot(-x) + (x^2+5x+2)\cdot 3$$

Distributive Property: $= x^2\cdot 2x^2 + 5x\cdot 2x^2 + 2\cdot 2x^2 + x^2\cdot(-x) + 5x\cdot(-x) + 2\cdot(-x) + x^2\cdot 3 + 5x\cdot 3 + 2\cdot 3$

$= 2x^4 + 10x^3 + 4x^2 - x^3 - 5x^2 - 2x + 3x^2 + 15x + 6$

Combine like terms: $= 2x^4 + 9x^3 + 2x^2 + 13x + 6$

Quick ✓

In Problems 12 and 13, find the product.

12. $(2y - 3)(y^2 + 4y + 5)$ **13.** $(z^2 - 3z + 2)(2z^2 + z + 6)$

12. $2y^3 + 5y^2 - 2y - 15$
13. $2z^4 - 5z^3 + 7z^2 - 16z + 12$

4 Multiply Special Products

Certain binomials have products that result in patterns. These products are called special products.

EXAMPLE 6

Products of the Form $(A - B)(A + B)$

Classroom Example 6
Find the product: $(n - 5)(n + 5)$
Answer: $n^2 - 25$

Find the product: $(x - 7)(x + 7)$

Solution
Use FOIL and obtain

$$\begin{aligned}(x - 7)(x + 7) &= \overset{F}{x \cdot x} + \overset{O}{x \cdot 7} - \overset{I}{7 \cdot x} - \overset{L}{7 \cdot 7}\\ &= x^2 + 7x - 7x - 49\\ &= x^2 - 49\end{aligned}$$

Work Smart
When products of the form $(A - B)(A + B)$ are multiplied, the "middle terms" will always be "opposites" and therefore sum to 0. Also, do not forget multiplication is commutative. Therefore, $(A + B)(A - B) = (A - B)(A + B) = A^2 - B^2$

The following rule is based on the results of Example 6.

Difference of Two Squares

$$(A - B)(A + B) = A^2 - B^2$$

EXAMPLE 7

Using the Difference of Two Squares Formula

Classroom Example 7
Find each product:
(a) $(2x + 5)(2x - 5)$
(b) $(4a - 7b^2)(4a + 7b^2)$
Answer: (a) $4x^2 - 25$
(b) $16a^2 - 49b^4$

(a)

$$\begin{aligned}(A + B)(A - B) &= A^2 - B^2\\ (3x + 2)(3x - 2) &= (3x)^2 - 2^2\\ &= 9x^2 - 4\end{aligned}$$

(b) $A = 2m$ and $B = 5n^2$

$$\begin{aligned}(2m - 5n^2)(2m + 5n^2) &= (2m)^2 - (5n^2)^2\\ &= 4m^2 - 25n^4\end{aligned}$$

Quick ✓

14. *True or False* The product of a binomial and a binomial is always a trinomial. False

15. $(A - B)(A + B) =$ $\underline{A^2 - B^2}$.

In Problems 16 and 17, find each product.

16. $(5y + 2)(5y - 2)$ $25y^2 - 4$ **17.** $(7y + 2z^3)(7y - 2z^3)$ $49y^2 - 4z^6$

Classroom Example 8
Find each product:
(a) $(3p + 4)^2$
(b) $(2x - 5y)^2$
Answer:
(a) $9p^2 + 24p + 16$
(b) $4x^2 - 20xy + 25y^2$

EXAMPLE 8

Products of the Form $(A + B)^2$ or $(A - B)^2$

Find each product:

(a) $(4x + 3)^2$ **(b)** $(3z - 5)^2$

Work Smart

$$(A + B)^2 \neq A^2 + B^2$$
$$(A - B)^2 \neq A^2 - B^2$$

Whenever you feel the urge to perform an operation that you're not quite sure about, try it with actual numbers. For example, does

$(3 + 2)^2 = 3^2 + 2^2$? NO!

So $(A + B)^2 \neq A^2 + B^2$

Solution

(a) $$(4x + 3)^2 = (4x + 3)(4x + 3) = (4x)^2 + \underbrace{4x \cdot 3 + 4x \cdot 3}_{= 2 \cdot 4x \cdot 3} + 3^2 = 16x^2 + 24x + 9$$

(b) $$(3z - 5)^2 = (3z - 5)(3z - 5) = (3z)^2 \underbrace{- 3z \cdot 5 - 3z \cdot 5}_{= -2 \cdot 3z \cdot 5} + 5^2 = 9z^2 - 30z + 25$$

Example 8 leads to some general results.

Squares of Binomials, or Perfect Square Trinomials

$$(A + B)^2 = A^2 + 2AB + B^2$$
$$(A - B)^2 = A^2 - 2AB + B^2$$

EXAMPLE 9 Using the Perfect Square Trinomial Formulas

Classroom Example 9

Find each product:
(a) $(n + 8)^2$
(b) $(7z - 2)^2$
(c) $(3x + 8y^2)^2$

Answer:
(a) $n^2 + 16n + 64$
(b) $49z^2 - 28z + 4$
(c) $9x^2 + 48xy^2 + 64y^4$

$$(A + B)^2 = A^2 + 2AB + B^2$$

(a) $$(w + 5)^2 = w^2 + 2 \cdot w \cdot 5 + 5^2 = w^2 + 10w + 25$$

$$(A - B)^2 = A^2 - 2AB + B^2$$

(b) $$(6p - 5)^2 = (6p)^2 - 2 \cdot 6p \cdot 5 + 5^2 = 36p^2 - 60p + 25$$

(c) $$(3x + 5y^2)^2 = (3x)^2 + 2 \cdot 3x \cdot 5y^2 + (5y^2)^2 = 9x^2 + 30xy^2 + 25y^4$$

Work Smart

If you cannot remember the formulas for a perfect square, do not panic! Use the fact that

$$(A + B)^2 = (A + B)(A + B)$$

and then FOIL. Use the same reasoning with perfect squares of the form $(A - B)^2$.

Quick ✓

18. $(A - B)^2 =$ $\underline{A^2 - 2AB + B^2}$; $(A + B)^2 =$ $\underline{A^2 + 2AB + B^2}$.

19. $x^2 + 2xy + y^2$ is referred to as a perfect square trinomial.

20. *True or False* $(x + a)^2 = x^2 + a^2$ False

In Problems 21 and 22, find each product.

21. $(z - 8)^2$ $z^2 - 16z + 64$ **22.** $(6p + 5)^2$ $36p^2 + 60p + 25$

R.10 Exercises

MyLabMath® Underlined exercise numbers have complete video solutions in MyLabMath or may be accessed using the QR code to the right.

Problems **1–22** *are the* **Quick ✓s** *that follow the* **EXAMPLES**.

Building Skills

In Problems 23–34, find the product. See Objective 1.

23. $(5xy^2)(-3x^2y^3)$ $-15x^3y^5$ **24.** $(9a^3b^2)(-3a^2b^5)$ $-27a^5b^7$

25. $\left(\frac{3}{4}yz^3\right)\left(\frac{20}{9}y^3z^2\right)$ $\frac{5}{3}y^4z^5$ **26.** $\left(\frac{12}{5}x^2y\right)\left(\frac{15}{4}x^4y^3\right)$ $9x^6y^4$

27. $5x(x^2 + 4x + 2)$ $5x^3 + 20x^2 + 10x$ **28.** $6y(y^2 - 4y + 3)$ $6y^3 - 24y^2 + 18y$

29. $-4a^2b(3a^2 + 2ab - b^2)$ $-12a^4b - 8a^3b^2 + 4a^2b^3$

30. $-3mn^3(4m^2 - mn + 5n^2)$ $-12m^3n^3 + 3m^2n^4 - 15mn^5$

31. $\frac{2}{3}ab\left(\frac{3}{4}a^2b - \frac{9}{8}ab^3 + 6ab\right)$ $\frac{1}{2}a^3b^2 - \frac{3}{4}a^2b^4 + 4a^2b^2$

32. $\frac{5}{2}xy\left(\frac{4}{15}x^2y - \frac{6}{5}xy + \frac{3}{10}xy^2\right)$ $\frac{2}{3}x^3y^2 - 3x^2y^2 + \frac{3}{4}x^2y^3$

33. $0.4x^2(1.2x^2 - 0.8x + 1.5)$ $0.48x^4 - 0.32x^3 + 0.6x^2$

34. $0.8y(0.4y^2 + 1.1y - 2.5)$ $0.32y^3 + 0.88y^2 - 2y$

In Problems 35–48, find the product of the two binomials. See Objective 2.

35. $(x + 3)(x + 5)$ $x^2 + 8x + 15$

36. $(y - 2)(y - 6)$ $y^2 - 8y + 12$

37. $(a + 5)(a - 3)$ $a^2 + 2a - 15$

38. $(z - 8)(z + 3)$ $z^2 - 5z - 24$

39. $(4a + 3)(3a - 1)$ $12a^2 + 5a - 3$

40. $(5x - 3)(x + 4)$ $5x^2 + 17x - 12$

41. $(4 - 5x)(3 + 2x)$ $-10x^2 - 7x + 12$

42. $(2 - 7y)(5 + 2y)$ $-14y^2 - 31y + 10$

43. $\left(\frac{2}{3}x + 2\right)\left(\frac{1}{2}x - 4\right)$ $\frac{1}{3}x^2 - \frac{5}{3}x - 8$

44. $\left(\frac{3}{2}y + 4\right)\left(\frac{4}{3}y - 1\right)$ $2y^2 + \frac{23}{6}y - 4$

45. $(4a + 3b)(a - 5b)$ $4a^2 - 17ab - 15b^2$

46. $(3m - 5n)(m + 2n)$ $3m^2 + mn - 10n^2$

47. $(2x^2 + 1)(x^2 - 3)$ $2x^4 - 5x^2 - 3$

48. $(3x^2 - 5)(x^2 + 2)$ $3x^4 + x^2 - 10$

In Problems 49–64, find the product of the polynomials. See Objective 3.

49. $(x + 1)(x^2 + 4x + 2)$ $x^3 + 5x^2 + 6x + 2$

50. $(y - 2)(y^2 + 5y - 3)$ $y^3 + 3y^2 - 13y + 6$

51. $(3a - 2)(2a^2 + a - 5)$ $6a^3 - a^2 - 17a + 10$

52. $(2b + 3)(3b^2 - 2b + 1)$ $6b^3 + 5b^2 - 4b + 3$

53. $(5z^2 + 3z + 2)(4z + 3)$ $20z^3 + 27z^2 + 17z + 6$

54. $(3p^2 - 5p + 3)(7p - 2)$ $21p^3 - 41p^2 + 31p - 6$

55. $-\frac{1}{2}x(2x + 6)(x - 3)$ $-x^3 + 9x$

56. $-\frac{4}{3}k(k + 7)(3k - 9)$ $-4k^3 - 16k^2 + 84k$

57. $(4 + y)(2y^2 - 3 + 5y)$ $2y^3 + 13y^2 + 17y - 12$

58. $(3 + 2z)(z^2 + 5 - 3z)$ $2z^3 - 3z^2 + z + 15$

59. $(w^2 + 2w + 1)(2w^2 - 3w + 1)$ $2w^4 + w^3 - 3w^2 - w + 1$

60. $(a^2 + 4a + 4)(3a^2 - a - 2)$ $3a^4 + 11a^3 + 6a^2 - 12a - 8$

61. $(b + 1)(b - 2)(b + 3)$ $b^3 + 2b^2 - 5b - 6$

62. $(2a - 1)(a + 4)(a + 1)$ $2a^3 + 9a^2 + 3a - 4$

63. $(2ab + 5)(4a^2 - 2ab + b^2)$ $8a^3b - 4a^2b^2 + 2ab^3 + 20a^2 - 10ab + 5b^2$

64. $(xy - 2)(x^2 + 2xy + 4y^2)$ $x^3y + 2x^2y^2 + 4xy^3 - 2x^2 - 4xy - 8y^2$

In Problems 65–78, find the special product. See Objective 4.

65. $(x - 6)(x + 6)$ $x^2 - 36$

66. $(y + 9)(y - 9)$ $y^2 - 81$

67. $(a + 8)^2$ $a^2 + 16a + 64$

68. $(b + 3)^2$ $b^2 + 6b + 9$

69. $(3y - 1)^2$ $9y^2 - 6y + 1$

70. $(4z - 5)^2$ $16z^2 - 40z + 25$

71. $(5a + 3b)(5a - 3b)$ $25a^2 - 9b^2$

72. $(8y + 3z)(8y - 3z)$ $64y^2 - 9z^2$

73. $(8z + y)^2$ $64z^2 + 16yz + y^2$

74. $(4a + 7b)^2$ $16a^2 + 56ab + 49b^2$

75. $(10x^2 - y)^2$ $100x^4 - 20x^2y + y^2$

76. $(7p - 3q^2)^2$ $49p^2 - 42pq^2 + 9q^4$

77. $(a^3 + 2b)(a^3 - 2b)$ $a^6 - 4b^2$

78. $(m^2 - 2n^3)(m^2 + 2n^3)$ $m^4 - 4n^6$

Mixed Practice

In Problems 79–100, simplify the expression.

79. $5ab(a - b)^2$ $5a^3b - 10a^2b^2 + 5ab^3$

80. $-3x(x - 3)^2$ $-3x^3 + 18x^2 - 27x$

81. $(x^2 + 3)(x^2 - 3)$ $x^4 - 9$

82. $(b^3 - 10)(b^3 + 10)$ $b^6 - 100$

83. $(z^2 + 9)(z - 3)(z + 3)$ $z^4 - 81$

84. $(y^2 + 4)(y + 2)(y - 2)$ $y^4 - 16$

85. $(2x^3 + 3)^2$ $4x^6 + 12x^3 + 9$

86. $(3a^4 - 2)^2$ $9a^8 - 12a^4 + 4$

87. $(2m - 3n)(4m + n) - (m - 2n)^2$ $7m^2 - 6mn - 7n^2$

88. $(6p + q)(5p - 2q) - (p + 3q)^2$ $29p^2 - 13pq - 11q^2$

89. $(x + 3)(x^2 - 3x + 9)$ $x^3 + 27$

90. $(2y + 3)(4y^2 - 6y + 9)$ $8y^3 + 27$

91. $\left(2x - \frac{1}{2}\right)^2$ $4x^2 - 2x + \frac{1}{4}$

92. $\left(3x + \frac{1}{3}\right)^2$ $9x^2 + 2x + \frac{1}{9}$

93. $(p + 2)^3$ $p^3 + 6p^2 + 12p + 8$

94. $(z - 3)^3$ $z^3 - 9z^2 + 27z - 27$

95. $(7x - 5y + 2)(3x - 2y + 1)$ $21x^2 - 29xy + 13x + 10y^2 - 9y + 2$

96. $(2a + b - 5)(4a - 2b + 1)$ $8a^2 - 18a - 2b^2 + 11b - 5$

97. $(2p - 1)(p + 3) + (p - 3)(p + 3)$ $3p^2 + 5p - 12$

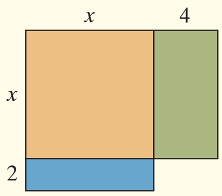
98. $(3z + 2)(z - 2) + (z + 2)(z - 2)$ $4z^2 - 4z - 8$

99. $(x + 3)(x - 3)(x^2 - 9) - (x + 1)(x^2 - 3)$ $x^4 - x^3 - 19x^2 + 3x + 84$

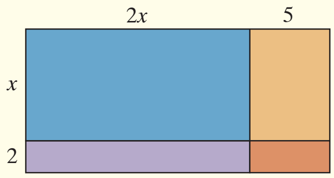
100. $(a + 2)(a - 2)(a^2 - 4) - (a + 3)(a^2 - 3)$ $a^4 - a^3 - 11a^2 + 3a + 25$

Applying the Concepts

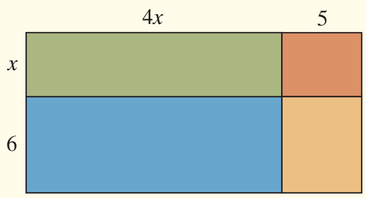
△ *The product of polynomials can be visualized by using the area of rectangles. In Problems 101–104, find a polynomial expression for the total area of each of the following figures.*

101. 

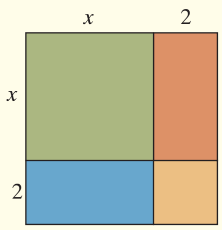
$x^2 + 6x$

102. 

$2x^2 + 9x + 10$

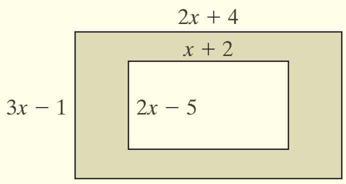
103. 

$4x^2 + 29x + 30$

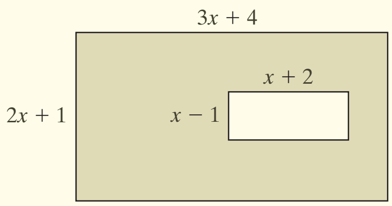
104. 

$x^2 + 4x + 4$

△ *In Problems 105 and 106, write a polynomial expression for the area of the shaded region of the figure.*

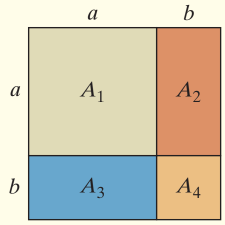
105. 

$4x^2 + 11x + 6$

106. 

$5x^2 + 10x + 6$

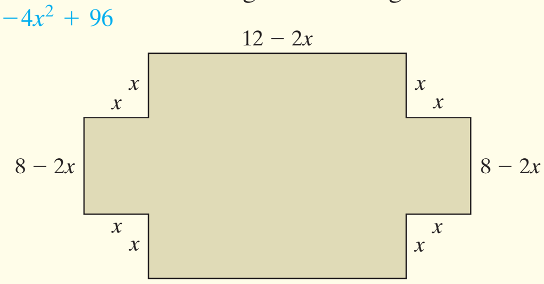
△ **107. Perfect Square** Why is the expression $(a + b)^2$ called a perfect square? Consider the figure below.



$A_1 = a^2; A_2 = ab;$ $A_3 = ab; A_4 = b^2$

(a) Find the area of each of the four quadrilaterals.
(b) Use the result from part (a) to find the area of the entire region. $a^2 + 2ab + b^2$
(c) Find the length and width of the entire region in terms of a and b. Use this result to find the area of the entire region. What do you notice?
$$A = (a + b)(a + b) = (a + b)^2 = a^2 + 2ab + b^2$$

△ **108. Area** Express as a polynomial in standard form the area of the shaded region in the figure shown.
$-4x^2 + 96$



In Problems 109–116, find the product.

109. $[3x - (y + 1)][3x + (y + 1)]$ $9x^2 - y^2 - 2y - 1$

110. $[5 - (a + b)][5 + (a + b)]$ $-a^2 - 2ab - b^2 + 25$

111. $[2a + (b - 3)]^2$ $4a^2 + 4ab - 12a + b^2 - 6b + 9$

112. $[(m + 4) - n]^2$ $m^2 + 8m - 2mn + n^2 - 8n + 16$

113. $(2^x + 3)(2^x - 4)$ $2^{2x} - 2^x - 12$

114. $(3^x - 1)(3^x - 9)$ $3^{2x} - 10(3^x) + 9$

115. $(5^y - 1)^2$ $5^{2y} - 2(5^y) + 1$

116. $(2^z - 4)^2$ $2^{2z} - 8(2^z) + 16$